THE VIRTUAL STRUCTURE PROBLEM FOR HIGHER MODULAR GROUPS

ROBYNN CORVELEYN, GEOFFREY JANSSENS, AND DORYAN TEMMERMAN

Abstract. We classify the finite groups G such that the finitely presented group $\mathcal{U}(\mathbb{Z}G)$ has the good property. Furthermore we obtain several characterisations in terms of properties of the simple factors of $\mathcal{U}(\mathbb{Q}G)$. Ring theoretically it is shown that it coincides with having only low-dimensional $\mathbb{Q}G$-components (i.e. at most 1×1 and exceptional 2×2 components). In particular, we solve a new instance of the virtual structure problem, generalising the free-by-free work. Cohomologically this happens if and only if all simple factors have virtual cohomological dimension a divisor of 4. Geometrically it is proven to be equivalent to the components acting discontinuously on $\mathbb{H}^5$. The latter properties are investigated for general lattices in semisimple algebraic $\mathbb{Q}$-groups of (inner) type A where in general the properties are no longer equivalent.

Contents

2020 *Mathematics Subject Classification.* 20G25,20E06, 20C05.

Key words and phrases. linear groups, amalgamated products, group rings, profinite density, Virtual structure problem.

The first author is grateful for financial support from the FWO and the F.R.S.–FNRS under the Excellence of Science (EOS) program (project ID 40007542). The second author is grateful to Fonds Wetenschappelijk Onderzoek Vlaanderen - FWO (grant 88258), and le Fonds de la Recherche Scientifique - FNRS (grant 1.B.239.22) for financial support.

TO DO list

Het volgende is een lijst van precieze te typen zaken die we al hebben geïdentificeerd (maar niet de heel kleine to do's die vaak in vorm van marginpar in het document staan):

(1) Make final form of the table for proof theorem 2.1 (Doryan maakt, Geof bepaalt wat erin)
(2) Write the preliminaries on SSP and exceptional components (Geof)
(3) Afwerken bewijs dat ($\mathrm{M}_{\mathrm{exc}}$) overgaat naar subnormale deelgroepen (Robynn)
(4) Karakter theorie sectie afwerken, i.e. schrijf een equivalent van proposition 2.9 voor het geval dat Q abels is en $\mathrm{cd}(G) = \{1, 2, 4\}$ (**Wie?**)
(5) ~~Restricties op A (elementary abelian 3 group) en index kernel (Robynn)~~
(6) Uitleg rond Block VSP en context (Geof)
(7) Bewijs proposition 3.3 (Geof)
(8) ~~Bewijs proposition 3.5 (welke simpele hebben vcd 4) (Robynn) + afwerken van het bewijs (Robynn en Doryan)~~
(9) Bewijs theorem 3.7 nu in finale vorm zetten (Geof)
(10) Bewijs Proposition 3.10 (Robynn)
(11) Wat the Zassenhaus property for semisimple algebra is, i.e. sectie 5.2 (Geof)
(12) bewijs proposition 5.1 (wie?)

De volgende zijn to do's die nog meer denkwerk zullen vereisen:

(i) Nadenken over verdere restricties in het niet-nilpotent geval
(ii) Bewijs vinden voor het nilpotent geval van Theorem 2.6
(iii) Beperkingen op de vorm van nilpotente groepen met ($\mathrm{M}_{\mathrm{exc}}$) vinden. Meer precies probeer een variant te vinden van Lemma 5.1.(6), Lemma 5.2.(3) en lemma 6.1 in de Free-by-Free paper.
(iv) Theorem 3.12 (band tussen ($\mathrm{M}_{\mathrm{exc}}$) en Discreet in $\mathrm{SL}_4(\mathbb{C})$) zien wat we net kunnen en willen zeggen
(v) De juiste geometric groep theoretische eigenschappen vinden ! (i.e. sectie 4)
(vi) Aantonen dat de exceptionele 2×2 de strong zassenhaus hebben.

1. Introduction

Blabla Kleinert

The Virtual structure problem for low degree.

Question 1.1 (Virtual Structure Problem)**.** Let $\mathcal{G}$ be a class of groups. Classify the finite groups G such that $\mathcal{U}(\mathbb{Z}G)$ has a subgroup of finite index lying in $\mathcal{G}$.

The aim of this article is to solve the above for the following class of groups

$$\mathcal{G}_{am} = \{\mathbb{Z}^n \times \prod_{i \in I} A_i \star_{C_i} B_i \mid [A_i : C_i], [B_i : C_i] \text{ are finite but not } 1\}.$$

The main bulk of the paper will be about classifying the finite groups G such that the only non-division algebra components of $\mathbb{Q}G$ are exceptional 2×2 components. Following result completes a line of research started more than 20 years ago with the papers ... .

Theorem 1.2. *Let G be a finite group. The following are equivalent*

(1) All simple components $\mathrm{M}_n(D)$, with $n \geq 2$, of $\mathbb{Q}G$ are exceptional components of type II, i.e. $n = 2$ and D isomorphic to $\{\mathbb{Q}, \mathbb{Q}(\sqrt{-d}), \left(\frac{-a,-b}{\mathbb{Q}}\right)\}$,
(2) $\mathrm{vcd}(\mathrm{SL}_1(\mathbb{Z}Ge)) \in \{0, 1, 2, 4\}$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$
(3) G is a quotient of one of the following families of groups : **blabla**

Moreover, in that case $\mathrm{SL}_1(\mathbb{Z}Ge)$ is discrete subgroup of $\mathrm{SL}_4(\mathbb{C})$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$. The converse holds iff **to complete**

Remark 1.3. • The fourth condition in Theorem 1.2 can also be equivalently stated that $\mathrm{SL}_1(\mathbb{Z}Ge)$ acts discontinuously on $\mathbb{H}_5$ or $\mathbb{H}_3$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$.

Or *PSL*?

- one can be more precise in the actual components that appear. Namely **put final list**

Denote by $\mathcal{C}(\mathbb{Q}G)$ the isomorphism types of the simple components of $\mathbb{Q}G$. Next, we record a list of geometric group theoretical properties of $\mathcal{U}(\mathbb{Z}G)$ that is equivalent to $\mathcal{C}(\mathbb{Q}G)$ to be as in Theorem 1.2.

Is het eerder $V(\mathbb{Z}G)$?

Theorem 1.4. *Let G be a finite group. The following are equivalent*
(1) $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_{am}$
(2) $\mathcal{U}(\mathbb{Z}G)$ is good
(3) Stuff Angel-Zalesski paper.

As a corollary we get a statement which appeared without proof in ??. For this recall the following two classes of groups

$$\mathcal{G}_{pab} = \{\prod_i A_{i,1} \star \cdots \star A_{i,t_i} \mid A_{i,j} \text{ are finitely generated abelian } \}$$

and

$$\mathcal{G}_{\neq 1} := \{\prod_i \Gamma_i \mid e(\Gamma_i) \neq 1\}.$$

The following was announced without proof in [??].

Corollary 1.5. *The following classes are equal*

$$\{G \text{ finite s.t. } \mathcal{U}(\mathbb{Z}G) \text{ is virtually-}\mathcal{G}_{\neq 1}\} = \{G \text{ finite s.t. } \mathcal{U}(\mathbb{Z}G) \text{ is virtually-}\mathcal{G}_{pab}\}.$$

Theorem 1.2 and its predecessors suggest to investigate the following ring theoretical variant of the virtual structure problem for $\mathcal{U}(\mathbb{Z}G)$. **put the statement that is now later in the paper**

How to call the problem?

Problem. Let $\mathcal{P}$ be a set of isomorphism classes of finite dimensional simple algebras over $\mathbb{Q}$. Classify all finite groups G such that $\mathcal{C}(\mathbb{Q}G) \subseteq \mathcal{P}$.

The blockwise Zassenhaus property. In the 70's Zassenhaus formulated a set of conjectures which had to clarify the origin of the conjectural isomorphism between two group bases. The strongest of these, called the Third Zassenhaus conjecture, asserted that every finite subgroup H of $V(\mathbb{Z}G)$ is conjugated over $\mathbb{Q}G$ to a subgroup of G. This has been disproven in [??]. It nevertheless an important problem to determine which classes of groups satisfy the property asserted by the conjecture.

We define the Zassenhaus property, with respect to a type of subgroups, for any semisimple algebra. Subsequently we consider what we call the blockwise Zassenhaus property. For this no counterexample is yet known and in this paper we prove the following.

Theorem 1.6. *Let G be a finite group and $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that $\mathbb{Q}Ge$ is a field, totally definite quaternion algebra or an exceptional simple algebra. Then $(\mathbb{Q}G)e$ satisfy the Third Zassenhaus property.*

Corollary 1.7. *If G is a finite group such that $\mathbb{Q}G$ XX. Then it satisfies the blockwise Zassenhaus property.*

To finish, we would like to advertise the study of the block-wise version of the Isomorphism problem and the Zassenhaus conjectures. It can namely be verified that the known counterexamples to those conjectures are not counterexamples to the block-wise version!

Acknowledgment. We thank Oberwolfach research in pairs (number) blabla. We are grateful to Angel del Río for sharing with us a proof of Lemma 5.2.

Conventions and notations. Throughout the full article G will denote a finite group. All orders will be understood to be $\mathbb{Z}$-orders. We also use the following notations:

- $\mathrm{PCI}(FG)$ for the set of primitive central idempotents of FG

- $\pi_e : \mathcal{U}(FG) \twoheadrightarrow FGe$ projection to a simple component
- $\mathcal{C}(FG) = \{FGe \mid e \in \mathrm{PCI}(FG)\}$ for the set of isomorphism types of the simple components of FG.
- Degree and index of a central simple algebra is **blabla**
- By $\phi(n)$ we denote Euler's phi function.

2. Finite groups with only exceptional higher simple components

DEFINE exceptional component + explain terminology (also which 1×1 are ok).

2.1. Preliminaries on describing simple component.

2.1. Preliminaries on describing simple component. Here put minimal necessary background on SSP theory. Hereby add as citation some of the main works on it (at least those containing the actual results we borrow from the book)

2.2. Restrictions on Division algebras and the Group. Now consider the following condition on G in terms of $\mathbb{Q}G$:

$$(\mathrm{M}_{\mathrm{exc}}) \quad \text{all } \mathbb{Q}Ge \cong \mathrm{M}_n(D), \text{ with } n \geq 2, \text{ are of the form } \mathrm{M}_2(\mathbb{Q}(\sqrt{-d})) \text{ or } \mathrm{M}_2\left(\left(\frac{-a,-b}{\mathbb{Q}}\right)\right) \text{ with } a, b, d \in \mathbb{N}.$$

maybe avoid this notation??

We will also use the notation $(\mathrm{M}_{\mathrm{exc}})^+$ to mean the stronger property that all non-division matrix components are of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \in \mathbb{N}$. The groups with $(\mathrm{M}_{\mathrm{exc}})^+$ and the extra assumption that the division components are all totally definite quaternion algebras were classified in [15].

Description components and first properties. In case of property $(\mathrm{M}_{\mathrm{exc}})$ however also exceptional division algebra components can arise as shown by the following result. Nevertheless the possible division algebra that can arise as $\mathbb{Q}Ge$ are still restricted.

Theorem 2.1. *Let G be a finite group having property $(\mathrm{M}_{\mathrm{exc}})$. Then*

(1) the 1×1 components of $\mathbb{Q}G$ are either fields or quaternion algebras. More precisely, the non-commutative possibilities are[1]*:*

$$\left\{\left(\frac{-1,-1}{\mathbb{Q}(\zeta_m)}\right), \left(\frac{-1,-3}{\mathbb{Q}}\right), \left(\frac{\zeta_{2^t},-3}{\mathbb{Q}(\zeta_{2^t})}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right) \mid m \in 2\mathbb{N}+1,\ t \in \mathbb{N}_{\geq 3}\right\}$$

(2) The only exceptional type II components are

$$\left\{\mathrm{M}_2(\mathbb{Q}(\sqrt{-d})), \mathrm{M}_2\left(\left(\frac{-1,-1}{\mathbb{Q}}\right)\right), \mathrm{M}_2\left(\left(\frac{-1,-3}{\mathbb{Q}}\right)\right) \mid d = 0,1,2,3\right\}.$$

Notatie veranderen, geen type II

(3) If $\mathbb{Q}Ge$ is exceptional type II, then $\pi(Ge) \subseteq \{2,3\}$.

(4) G has[2] *an abelian normal subgroup A with $\exp(G/A) \mid 4$.*

In particular, G is metabelian with $\mathrm{cd}(G) \subseteq \{1,2,4\}$. Furthermore, $\deg(\mathbb{Q}Ge) \mid 4$ for every $e \in \mathrm{PCI}(\mathbb{Q}G)$.

say that later some parts will be made stronger.

Remark 2.2. In the proof we will obtain that $C_3 \rtimes C_{2^n}$, where the action is by inversion, the rational group algebra has all non-division simple components of the form $\mathrm{M}_2(\mathbb{Q})$ and $\mathrm{M}_2(\mathbb{Q}(i))$. Furthermore it has a division component of the form $\left(\frac{\zeta_{2^{n-1}},-3}{\mathbb{Q}(\zeta_{2^{n-1}})}\right)$. Hence it gives an example of a group satisfying $(\mathrm{M}_{\mathrm{exc}})^+$ but not the stronger property considered in [15].

The proof of Theorem 2.1 will go through understanding the quotient groups Ge for $e \in \mathrm{PCI}(\mathbb{Q}G)$.

[1] Note that the only exceptional component of type II not appearing is $\mathrm{M}_2(\mathbb{H}_5)$.

[2] The proof will furthermore prove that $[Ge : Ae] \mid 4$ for every $e \in \mathrm{PCI}(\mathbb{Q}G)$.

Lemma 2.3. *For any finite group G, normal subgroup $N \leqslant G$ and $e \in \mathrm{PCI}(\mathbb{Q}G)$ holds:*

$$\mathcal{C}(\mathbb{Q}[G/N]) \subseteq \mathcal{C}(\mathbb{Q}G) \text{ and } \mathcal{C}(\mathbb{Q}[Ge]) \subseteq \mathcal{C}(\mathbb{Q}G).$$

Consequently,

$$\mathcal{C}(\mathbb{Q}G) = \bigcup_{e \in \mathrm{PCI}(\mathbb{Q}G)} \mathcal{C}(\mathbb{Q}[Ge]).$$

In particular, property $(\mathrm{M}_{\mathrm{exc}})$ *is inherited by quotients.*

Proof. The first claim follows from the fact that $\mathbb{Q}[G/N]$ is a semisimple subalgebra of $\mathbb{Q}G$. Indeed, it is immediately semisimple (since it is a group algebra), and a straightforward calculation shows that $\mathbb{Q}[G/N] \cong \mathbb{Q}G\widetilde{N} \leq \mathbb{Q}G$ with $\widetilde{N}$ the central idempotent $\frac{1}{|N|}\sum_{n \in N} n$. The second inclusion follows from the first since the group Ge is an epimorphic image of G. The rest is now also a direct consequence as every simple component of $\mathbb{Q}G$ corresponds to a primitive central idempotent $e \in \mathrm{PCI}(\mathbb{Q}G)$. □

Using Lemma 2.3, the proof of Theorem 2.1 reduces to a study of the fixed-point free groups classified by Amitsur [1] and the finite subgroups of exceptional components classified in [4]. In fact the conclusion of Theorem 2.1 already holds under the weaker condition that each Ge is embedded in a division algebra or an exceptional type II algebra.

Proof of Theorem 2.1. For a group G having $(\mathrm{M}_{\mathrm{exc}})$, the set $\mathrm{PCI}(\mathbb{Q}G)$ naturally decomposes into $\mathrm{PCI}_1 := \{e \mid \mathbb{Q}Ge \cong D\}$ and $\mathrm{PCI}_2 := \{e \mid \mathbb{Q}Ge \cong \mathrm{M}_2(D)\}$ where D always signifies a rational division algebra. Hence, with Lemma 2.3 in mind, for the first statement it suffices to analyse the components possibly appearing in $\mathbb{Q}[Ge]$ for $e \in \mathrm{PCI}_1$ or PCI_2.

Let's start with PCI_2. The finite subgroups $\mathcal{G}$ of $\mathrm{GL}_2(\mathbb{Q}(\sqrt{-d}))$ or $\mathrm{GL}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ with $a, b, d \in \mathbb{N}$ with the property that $\mathrm{span}_{\mathbb{Q}}(\mathcal{G})$ is the respective $\mathrm{M}_2(\cdot)$ have been classified[3] in [4, Theorem 3.7]. This classification consists of 55 groups and in particular the groups Ge for $e \in \mathrm{PCI}_2$ must be among these. One can compute the simple components for example in GAP using the Wedderga package, see table ?? in Appendix ?? for the result. This would moreover show that for groups Ge with $e \in \mathrm{PCI}_2$ the property $(\mathrm{M}_{\mathrm{exc}})$ is equivalent to the weaker property that each non-division component has reduced degree 2. A case-by-case verification also shows that each of these groups Ge contain an abelian normal subgroup of index a divisor of 4. Furthermore, as written in the table the only 1×1 components appearing are Still to add the table!

$$\mathbb{Q}, \mathbb{Q}(\zeta_3), \mathbb{Q}(i), \mathbb{Q}(\zeta_8), \mathbb{Q}(\zeta_{12}), \left(\frac{-1,-1}{\mathbb{Q}}\right), \left(\frac{-1,-3}{\mathbb{Q}}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})}\right).$$

Inspection of the table also shows that all other statements hold for such groups.

Next we consider the case that $e \in \mathrm{PCI}_1$ for which a similar reasoning applies. Indeed, the finite subgroups (such as Ge for $e \in \mathrm{PCI}_1$) of rational division algebras have been classified by Amitsur in [1]. We will use the rephrasing from [21, Theorem 2.1.4] which asserts that they are:

a) a **Z**-group, i.e. a subgroup of a rational division algebra with cyclic Sylow-subgroups.
b) i) the binary octahedral group O^* of order 48:

$$\{\pm 1, \pm i, \pm j, \pm ij, \frac{\pm 1 \pm i \pm j \pm ij}{2}\} \cup \{\frac{\pm a \pm b}{\sqrt{2}} \mid a, b \in \{1, i, j, ij\}\}.$$

ii) $C_m \rtimes Q$, where m is odd, Q is quaternion of order 2^t for some $t \geq 3$, an element of order 2^{t-1} centralizes C_m and an element of order 4 inverts C_m.
iii) $M \times Q_8$, with M a **Z**-group of odd order m and the (multiplicative) order of 2 mod m is odd.

[3] In [4, Table 2] a group was missing, see [3, Appendix A] for a complete list.

iv) $M \times \mathrm{SL}_2(\mathbb{F}_3)$, where M is a **Z**-group of order m coprime to 6 and the (multiplicative) order of 2 mod m is odd.

c) $\mathrm{SL}_2(\mathbb{F}_5)$.

Neither $O^*, \mathrm{SL}_2(\mathbb{F}_3)$ nor $\mathrm{SL}_2(\mathbb{F}_5)$ have $(\mathrm{M}_{\text{exc}})$. Indeed using[4] the Wedderga package in Gap one learns that $\mathrm{M}_3(\mathbb{Q})$ is a simple component over $\mathbb{Q}$ of $O^* \cong \mathrm{SU}_2(\mathbb{F}_3)$ and $\mathrm{SL}_2(\mathbb{F}_3)$ and $\mathrm{M}_5(\mathbb{Q})$ for $\mathrm{SL}_2(\mathbb{F}_5)$. Consequently, they can not be epimorphic images of the group G. Hence the cases b) i), iv) and c) do not appear as groups Ge for $e \in \mathrm{PCI}_1$.

The groups in b) ii) are actually dicyclic groups of order $2^t m$, i.e. Dic_{4n} with $n = 2^{t-2}m$ and $t \geq 3$. The case of odd n will be case (b) in the family of **Z**-groups. Therefore consider a general dicyclic group:

$$Dic_{4n} = \langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle.$$

Being a metabelian group its Strong Shoda pairs are described by [13, Theorem 3.5.12] which we apply now. Note that the commutator subgroup $Dic'_{4n} = \langle a^2 \rangle$ and $\langle a \rangle$ is the maximal abelian containing it. For any (H, K) SSP holds that $\langle a \rangle \subseteq H$. In other words $H = \langle a \rangle$ or Dic_{4n}. If $H = Dic_{4n}$ then the simple component associated to (H, K) is a field, [12, Lemma 2.4]. Via [13, Theorem 3.5.12] it is a direct verification that for $d \mid 2n$ the tuple $(\langle a \rangle, \langle a^d \rangle)$ is a SSP if and only if $d \neq 1, 2$. Note that K is normal in Dic_{4n}, hence the associated primitive central idempotent is $\epsilon(\langle a \rangle, \langle a^d \rangle)$. Now, in [13, Example 3.5.7], it is noted that

$$\mathbb{Q}Dic_{4n}\epsilon(\langle a \rangle, \langle a^d \rangle) \cong \mathrm{M}_2(\mathbb{Q}(\Re(\zeta_d))) \text{ if } d \mid n \text{ and } d \nmid 2 \tag{2.1}$$

where ζ_d denotes a complex primitive d-th root of unity. Since $\Re(\zeta_d) = \cos\frac{2\pi}{d} \in \mathbb{R} \setminus \mathbb{Q}$, Niven's theorem tells that $\mathrm{M}_2(\mathbb{Q}(\Re(\zeta_d)))$ is exceptional[5] if and only if $\varphi(d) \leq 2$. The latter is equivalent to $d \in \{1, 2, 3, 4, 6\}$. In conclusion if Dic_{4n} has $(\mathrm{M}_{\text{exc}})$, then it must be isomorphic to $Q_8, Q_{16}, C_3 \rtimes Q_8$ or $C_3 \rtimes C_4$. The first three groups are in the family b) ii). Moreover these groups indeed have $(\mathrm{M}_{\text{exc}})$. Precisely:

$$\mathcal{C}(\mathbb{Q}Q_8) = \{\mathbb{Q}, \left(\frac{-1,-1}{\mathbb{Q}}\right)\}, \qquad \mathcal{C}(\mathbb{Q}Q_{16}) = \{\mathbb{Q}, \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})}\right), \mathrm{M}_2(\mathbb{Q})\} \text{ and}$$
$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes Q_8]) = \{\mathbb{Q}, \left(\frac{-1,-1}{\mathbb{Q}}\right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right), \mathrm{M}_2(\mathbb{Q})\}$$

Before we consider case b) iii), we will discuss a), the **Z**-groups.

The **Z**-groups themselves have also been classified, see [21, Theorem 2.1.5]. They are the following:

a) cyclic.

b) $C_m \rtimes C_4$, where m is odd and C_4 acts by inversion.

c) $G_0 \times G_1 \times \ldots \times G_s$, with $s \geq 1$, the orders of the groups G_i are coprime and G_0 is the only cyclic subgroup amongst them. Each of the G_i, for $1 \leq i \leq s$, is of the form

$$C_{p^a} \rtimes \left(C_{q_1^{b_1}} \times \ldots \times C_{q_r^{b_r}}\right),$$

for $p, q_1, \ldots, q_r$ distinct primes. Moreover, each of the groups $C_{p^a} \rtimes C_{q_j^{b_j}}$ is non-cyclic (i.e. if $C_{q^{\alpha_j}}$ denotes the kernel of the action of $C_{q_j^{b_j}}$ on C_{p^a}, then $\alpha_j \neq b_j$) and satisfies the following properties:

(i) $q_j o_{q_j^{\alpha_j}}(p) \nmid o_{\frac{|G|}{|G_i|}}(p)$.

[4]The SmallGroup ID of the three groups are respectively [48,28], [24,3] and [120,5]

[5]Recall that $[\mathbb{Q}(\Re(\zeta_d)) : \mathbb{Q}] = \varphi(d)/2$. In particular, in contrast to the case $e \in \mathrm{PCI}_2$, it can happen that all non-division components are exceptional without the group having $(\mathrm{M}_{\text{exc}})$. Even more it can happen that all non-division components are of the form $\mathrm{M}_2(F)$ with F a quadratic extension of $\mathbb{Q}$. As shown by (2.1) this namely holds for Dic_{4n} with $n = 5, 10, 8, 12$.

(ii) one of the following is true:

- $q_j = 2, p \equiv -1 \mod 4,$ and $\alpha_j = 1$,
- $q_j = 2, p \equiv -1 \mod 4,$ and $2^{\alpha_j+1} \nmid p^2 - 1$,
- $q_j = 2, p \equiv 1 \mod 4,$ and $2^{\alpha_j+1} \nmid p - 1$,
- $q_j > 2,$ and $q_j^{\alpha_j+1} \nmid p - 1$.

It is clear the cyclic groups have ($\mathrm{M}_{\mathrm{exc}}$) since $\mathbb{Q}C_n$ is abelian. Moreover, by the well known theorem of Perlis-Walker, $\mathcal{C}(\mathbb{Q}C_n) = \{\mathbb{Q}(\zeta_d) \mid d \text{ divides } n\}$.

Case b), i.e Dic_{4n} with n odd, was already handled via (2.1). The conclusion was that the only possible (non-abelian) such group having ($\mathrm{M}_{\mathrm{exc}}$) is $C_3 \rtimes C_4$. In this case

$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes C_4])) = \{\mathbb{Q}, \mathbb{Q}(i), \left(\frac{-1,-3}{\mathbb{Q}}\right), \mathrm{M}_2(\mathbb{Q})\}.$$

Next consider case c). We first show that ($\mathrm{M}_{\mathrm{exc}}$) enforces $2 \mid |G_i|$, for $1 \leq i \leq s$ and hence $s = 1$ by the coprime condition. For this consider $A_i = \prod_{j=1}^{s} C_{q^{\alpha_j}}$, the kernel of the action. Then $B := G_i/A_i \cong C_{p^a} \rtimes C_{q_1^{k_1}.\cdots.q_s^{k_s}}$ where $k_j = b_j - \alpha_j > 0$ and the action is non-trivial and faithful. Denote $B = \langle x \rangle \rtimes \langle y \rangle$. By Lemma 2.3 the group B also has ($\mathrm{M}_{\mathrm{exc}}$). Note that $C_{p^a} = \langle x \rangle$ is a maximal abelian subgroup of B containing B'. Now using [13, Theorem 3.5.12] it is a direct verification that $(H, K) = (\langle x \rangle, 1)$ is a SSP of B. Moreover $\mathbb{Q}Be(G, \langle x \rangle, 1) \cong \mathbb{Q}(\zeta_{o(x)}) * \langle y \rangle$ for some explicit crossing (see [13, Remark 3.5.6]) which imply that the component is non-division. Now using [10, Lemma 3.4], we compute that

$$\dim_{\mathbb{Q}} \mathbb{Q}Be(G, \langle x \rangle, 1) = [G : \langle x \rangle]\phi(o(x)) = q_1^{k_1}.\cdots.q_s^{k_s} p^{a-1}(p-1).$$

On the other $\dim_{\mathbb{Q}} \mathbb{Q}Be(G, \langle x \rangle, 1) \mid 16$ as B has ($\mathrm{M}_{\mathrm{exc}}$). Combining both with the fact that p and the q_i are different primes, we obtain that $s = 1, q_1 = 2$ and $p^a = 3$. Thus $B \cong C_3 \rtimes C_{2^{k_1}}$. Furthermore, as the action is faithful, we obtain that $k_1 = 1$, i.e. $B \cong C_3 \rtimes C_2$ where the action is by inversion. Consequently $G_1 \cong C_3 \rtimes C_{2^{b_1}}$ with the action being inversion (as $\alpha_1 = b_1 - 1$).

It remains to consider $G_0 \times G_1$. As $\pi(G_1) = \{2, 3\}$ we have that $G_0 \cong C_m$ with m relatively prime to 2 and 3. But $\mathbb{Q}[G_0 \times G_1]$ contains as a simple component $\mathbb{Q}(\zeta_m) \otimes_{\mathbb{Q}} \mathrm{M}_2(\mathbb{Q}(\sqrt{-d})) \cong \mathrm{M}_2(\mathbb{Q}(\zeta_m)) \oplus \mathrm{M}_2(\mathbb{Q}(\zeta_m, \sqrt{-d}))$ with $d \in \mathbb{N}$ (potentially zero). As $G_0 \times G_1$ is assumed to have ($\mathrm{M}_{\mathrm{exc}}$) this implies that $[\mathbb{Q}(\zeta_m) : \mathbb{Q}] \leq 2$. The latter happens exactly when $m \mid 6$, which by the restriction on m ony happens for $m = 1$. Thus in conclusion[6] the only groups of

type c) with ($\mathrm{M}_{\mathrm{exc}}$) are those of the form $C_3 \rtimes C_{2^n}$ with the action being by inversion.

For such groups, using [13, Theorem 3.5.12], one can verify that for $n \geq 4$

$$\mathcal{C}(\mathbb{Q}[C_3 \rtimes C_{2^n}]) = \{\mathbb{Q}(\zeta_{2^\ell}), \left(\frac{-1,-3}{\mathbb{Q}}\right), \left(\frac{\zeta_{2^t},-3}{\mathbb{Q}(\zeta_{2^t})}\right), \mathrm{M}_2(\mathbb{Q}), \mathrm{M}_2(\mathbb{Q}(i)) \mid 1 \leq \ell \leq n,\ 3 \leq t \leq n-1\}.$$

The last case to handle is b) iii), i.e. $M \times Q_8$ with M a **Z**-group of odd order m and also the multiplicative order of 2 modulo m is odd. By Lemma 2.3 also M has ($\mathrm{M}_{\mathrm{exc}}$), but looking at the possible such **Z**-groups of odd order we see that[7] M must be cyclic. Now recall that $\mathcal{C}(\mathbb{Q}Q_8) = \{\mathbb{Q}, \left(\frac{-1,-1}{\mathbb{Q}}\right)\}$. If M is cyclic, then $\mathcal{C}(\mathbb{Q}[M \times Q_8]) = \{\mathbb{Q}(\zeta_d), \left(\frac{-1,-1}{\mathbb{Q}(\zeta_d)}\right) \mid d \text{ divides } m\}$. As m is odd all the components are division algebras, a conclusion that also directly would have followed from [20]. In particular the groups

Klopt dit? Voor $d = 3$ splits het bv denk ik

$$M \times Q_8 \text{ with } m \text{ and } o_m(2) \text{ odd have } (\mathrm{M}_{\mathrm{exc}}).$$

[6] To reach this conclusion we only used that every non-division component is of the form $\mathrm{M}_2(D)$ with $[\mathcal{Z}(D) = \mathbb{Q}] \leq 2$.

[7] Also this conclusion only requires that every non-division component is of the form $\mathrm{M}_2(D)$ with $[\mathcal{Z}(D) = \mathbb{Q}] \leq 2$ and not the full strength of ($\mathrm{M}_{\mathrm{exc}}$).

To summarise, with the analysis above we have shown part (1) and (2) from the statement by describing $\prod_{e\in \mathrm{PCI}(\mathbb{Q}G)} Ge$. Note that all allowed groups Ge have been highlighted in the proof. We see that they all have an abelian normal subgroup A_e with $[Ge : A_e] \mid 4$. Hence $A = G \cap \prod_{e\in \mathrm{PCI}(\mathbb{Q}G)} A_e$ is an abelian normal subgroup of G with $\exp(G/A) \mid 4$. Consequently, G is metabelian.

Finally, for the simple algebras $\mathrm{M}_n(D)$ allowed by (M_{exc}), we see that $\mathrm{M}_2(D) \otimes_{\mathbb{Q}} \mathbb{C}$ is isomorphic to $\mathrm{M}_2(\mathbb{C}) \oplus \mathrm{M}_2(\mathbb{C})$ or $\mathrm{M}_4(\mathbb{C})$. By the first part if $\mathbb{Q}Ge \cong D$, then $D \otimes_{\mathbb{Z}(D)} \mathbb{C}$ is either $\mathbb{C}$ or $\mathrm{M}_2(\mathbb{C})$. So indeed $\mathrm{cd}(G) \subseteq \{1, 2, 4\}$. □

Determine where the SSP lemma should come (i think characterisation section) The aim of the remaining of the section is to obtain more precise descriptions of the groups having (M_{exc}) . To start we study $\mathrm{PCI}(\mathbb{Q}G)$. Recall that G is called *strongly monomial* if each primitive central idempotent e of $\mathbb{Q}G$ comes from a SSP, i.e. $e = e(G, H, K)$ for some SSP (H, K). For example all abelian-by-supersolvable groups are strongly monomial [13, Theorem 3.5.10]. In particular, by Theorem 2.1, (M_{exc}) implies strongly monomial.

Say something about which of the cases give allowed components. Maybe $[H : K] = 8$ or 12 not possible?

Lemma 2.4. *Suppose G has (M_{exc}) and let (H, K) be a SSP of G with*[8] *$H \neq G$. Denote $e = e(G, H, K)$ the associated primitive central idempotent. Then $[G : H] \mid 4$. Furthermore the following holds:*

(1) if $N_G(K) = H$, then[9] *$\phi([H : K]) = \dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) \leq 2$ and $[G : H] = 2$,*
(2) if $H \lneq N_G(K) \lneq G$, then $[H : K] \in \{3, 4, 6\}$ and $[G : H] = 4$,
(3) if $N_G(K) = G$ and $\mathbb{Q}Ge$ is not a division algebra, then

$$\phi([H : K]) = 2 \dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) \in \{2, 4\} \text{ and } [H : K] \mid 8 \text{ or } 12$$

if $[G : H] = 2$ and $[H : K] = 8$ or 12 if $[G : H] = 4$.

Proof. In general $\mathbb{Q}Ge(G, H, K) \cong \mathrm{M}_{[G:N_G(K)]}(\mathbb{Q}(\zeta_{[H:K]}) * N_G(K)/H)$ for some explicit crossing. Furthermore $\deg(\mathbb{Q}Ge(G, H, K)) = [G : H]$ where 'deg' denotes the degree of the central simple algebra $\mathbb{Q}Ge(G, H, K))$ (i.e. $\deg(A) = \sqrt{n}$ if $A \otimes_{\mathcal{Z}(A)} \mathbb{C} \cong \mathrm{M}_n(\mathbb{C})$). It follows from the description of the possible simple components in Theorem 2.1 that $[G : H] = \deg(\mathbb{Q}Ge(G, H, K))) \mid 4$.

For the other parts of the statement we make an analysis of the various cases. We will denote $N := N_G(K)$.

If $N = H$, then $\mathbb{Q}Ge \cong \mathrm{M}_{[G:H]}(\mathbb{Q}(\zeta_{[H:K]}))$. As G has (M_{exc}) and $G \neq H$ this means that $[G : H] = 2$ and $[\mathbb{Q}(\zeta_{[H:K]}) : \mathbb{Q}] \leq 2$. In other words, $\dim_{\mathbb{Q}} \mathbb{Q}Ge = [\mathbb{Q}(\zeta_{[H:K]}) : \mathbb{Q}] = \phi([H : K]) \leq 2$, as stated.

Now suppose that $H \lneq N \lneq G$. As $[G : H] \mid 4$, the condition entails that $[G : N] = 2 = [N : H]$. Consequently $\mathbb{Q}Ge \cong \mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ for some $a, b \in \mathbb{N}_0$, as it is the only simple component of degree 4 in case G has (M_{exc}). Using the dimension formula we obtain that

$$16 = \dim_{\mathbb{Q}} \mathbb{Q}Ge = 4.2.\phi([H : K]).$$

Thus $\phi([H : K]) = 2$ and hence $[H : K] = 3, 4$ or 6.

Finally suppose that K is normal in G. Then $\mathbb{Q}Ge \cong \mathbb{Q}(\zeta_{[H:K]}) * G/H$. Hence if it is not a division algebra, then it is an exceptional algebra of type II. First consider the case that $[G : H] = 2$ and so $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \in \mathbb{N}$. Then $4 \dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) = \dim_{\mathbb{Q}} \mathbb{Q}Ge = 2.1.\phi([H : K])$, so indeed $\phi([H : K]) = 2 \dim_{\mathbb{Q}} \mathcal{Z}(\mathbb{Q}Ge) \in \{2, 4\}$. It remains to consider the case that $[G : H] = 4$ and so $\mathbb{Q}Ge \cong \mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ for some $a, b \in \mathbb{N}_0$. The dimension formula now gives $16 = 4.1.\phi([H : K])$, i.e. $\phi([H : K]) = 4$.

Now $\phi([H : K]) = 4$ is equivalent to $[H : K] \in \{5, 8, 10, 12\}$. We claim that $5 \nmid [H : K]$. Indeed, as $\ker(\varphi_e : G \to Ge) = core_G(K) \leq K$ one has that $[H : K] = [\varphi_e(H) : \varphi_e(K)]$.

[8]The condition $H = G$ is equivalent to $G' \leq K$ which in turn is equivalent to that $\mathbb{Q}Ge(G, H, K)$ is non-commutative, [12, Lemma 2.4].

[9]In other words $[H : K]$ divides 4 or 6 if $N_G(K) = H$ and it divides 8 or 12 if $N_G(K) = G$.

Thanks to Theorem 2.1, $\pi(G) \subseteq \{2,3\}$ and thus $5 \nmid [\varphi_e(H) : \varphi_e(K)]$, yielding the claim. In conclusion, $[H : K] = 8$ or 12 when $\phi([H : K]) = 4$, finishing the proof. □

For the next technical result we consider the set

$$\mathcal{E} := \{e \in \mathrm{PCI}(\mathbb{Q}G) \mid \mathbb{Q}Ge \text{ not exceptional type I }\}.$$

Lemma 2.5. *Suppose G has* $(\mathrm{M}_{\mathrm{exc}})$ *and let $f := \sum_{e\in\mathcal{E}} e$. Then $G \cong A \rtimes Q$ with A an abelian group of odd order and Q a 2-group. Furthermore,*

(1) $\exp(Gf) \mid \ldots$ *and* $\exp(\mathcal{Z}(Gf)) \mid \ldots$
(2) $\mathcal{Z}(G) \cap G' = \ldots$
(3) if G non-nilpotent then $\exp(G_2) \mid 8$ *and if $A \neq 1$, then... (see lemma 6.1 in free-by-free)*

Note that $f = 1$ if $\mathcal{E} = \mathrm{PCI}(\mathbb{Q}G)$, i.e. when $\mathbb{Q}G$ has no simple component which is a division algebra different of a field or a totally definite quaternion algebra. This is for example the case when G is a *cut group* [3, Proposition 6.12]. Recall that a group is called cut if $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite. It was shown in [??] that cut groups are exactly the inverse semi-rational groups.

Proof. To DO and Determine what to include/poursuivre... □

Some Nice index 2 *Subgroup.* All finite groups such that $\mathrm{SL}_1(\mathbb{Q}Ge)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ have been classified in [15]. In loc.cit. presentations for such groups were even given. Furthermore, they showed that the aforementioned property is equivalent to saying that all simple components of $\mathbb{Q}G$ are either fields, totally definite quaternion algebras or of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \geq 0$. The following result shows that groups with $(\mathrm{M}_{\mathrm{exc}})$ are index 2 overgroups of the groups classified in [15].

Theorem 2.6. *If $\mathbb{Q}G$ has* $(\mathrm{M}_{\mathrm{exc}})$ *then G has an index two subgroup H whose non-division components are all of the form* $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ *with $d \geq 0$.*

Is het waar dat H ook effectief minstens 1 niet-division component heeft? I.e. hebben we $\mathrm{cd}(H) = \{1, 2\}$?

Example 2.7. The converse of Theorem 2.6 is not true in general. For example consider the following extraspecial group of order 2^5 (whose SmallGroup ID is [32,49]):

$$D_8 \circ D_8 := \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, c^2 = a^2, a^b = a^{-1}, c^d = a^2c, \\ [a, c] = [a, d] = [b, c] = [b, d] = 1\rangle.$$

It can be verified that $\mathcal{C}(\mathbb{Q}[D_8 \circ D_8]) = \{\mathbb{Q}, \mathrm{M}_4(\mathbb{Q})\}$ (e.g. via the well-known description of its complex irreducible representations). On the other, e.g. to be proven via a manual check (via GAP), all the 2-groups until order 16, except D_{16}, have only matrix components of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \geq 0$.

To prove Theorem 2.6, we first need the following result of independent interest which will also be instrumental in Section 2.4.

Lemma 2.8. *If $\mathbb{Q}G$ has* $(\mathrm{M}_{\mathrm{exc}})$*, then so does $\mathbb{Q}H$, for any subnormal subgroup H of G.*

Proof. It suffices to prove that $(\mathrm{M}_{\mathrm{exc}})$ is inherited by any normal subgroup H of G, since then a recursive argument finishes the proof.

Consider a simple $\mathbb{Q}H$-module N, and decompose the induced $\mathbb{Q}G$-module $\mathrm{Ind}_H^G(N) = M_1 \oplus \ldots \oplus M_t$ into simple $\mathbb{Q}G$-modules. Then N is a direct summand of the restriction $\mathrm{Res}_H^G \mathrm{Ind}_H^G(N)$, and hence of one of the $\mathbb{Q}H$-modules $\mathrm{Res}_H^G(M_i)$. Therefore, in order to describe the simple $\mathbb{Q}H$-components of a $\mathbb{Q}G$-module, it suffices to investigate the modules $\mathrm{Res}_H^G(M)$ with M a simple $\mathbb{Q}G$-module. In the case at hand, we are interested in describing the simple $\mathbb{Q}H$-components of the $\mathbb{Q}H$-module $\mathrm{Res}_H^G(\mathbb{Q}G)$ obtained from the regular $\mathbb{Q}G$-module. The simple $\mathbb{Q}G$-components M of $\mathbb{Q}G$ are columns of matrix algebras $\mathrm{M}_n(D)$, and since $\mathbb{Q}G$ has $(\mathrm{M}_{\mathrm{exc}})$ by assumption, the form of $\mathrm{M}_n(D)$ is described by Theorem 2.1.

We recall Clifford's theorem, **[cite]**, which, since H is normal in G, implies that M decomposes as a $\mathbb{Q}H$-module into a direct sum

$$\mathrm{Res}_H^G(M) \simeq N_1 \oplus \cdots \oplus N_r$$

of simple $\mathbb{Q}H$-modules N_i which are all in the same G-orbit. Thus for all $1 \leq i, j \leq r$, $N_i = g \cdot N_j$ for some $g \in G$. In particular all N_i have the same $\mathbb{Q}$-dimension. Each N_i is a column of a matrix algebra $\mathrm{M}_{n_i}(D_i)$, with $\mathrm{M}_{n_i}(D_i)$ a component of $\mathbb{Q}H$. We investigate the N_i appearing in the above decomposition of M, for each possible simple $\mathbb{Q}G$-component M.

To be continued ☺ □

Proof of Theorem 2.6. By Theorem 2.1, G has character degrees $\mathrm{cd}(G) \subseteq \{1, 2, 4\}$. If G is non-nilpotent, then by [8, Theorem 1.1], either G has a subgroup H of index 2 such that $\mathrm{cd}(H) \subseteq \{1, 2\}$, or $G/\mathcal{Z}(G) \cong (C_3 \rtimes C_2) \wr C_2$. But a calculation using GAP, shows immediately that $\mathbb{Q}((C_3 \rtimes C_2) \wr C_2)$ does not have the $(\mathrm{M}_{\mathrm{exc}})$ property, and in particular this would imply by Lemma 2.3 that $\mathbb{Q}G$ does not have $(\mathrm{M}_{\mathrm{exc}})$, which is a contradiction with the assumption. Hence G has a subgroup H of index two with $\mathrm{cd}(H) \subseteq \{1, 2\}$. Now since H is normal, Lemma 2.8 implies that $\mathbb{Q}H$ has $(\mathrm{M}_{\mathrm{exc}})$. If $\mathbb{Q}H$ had a component of the form $\mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$, this would imply[10] that $4 \in \mathrm{cd}(H)$, a contradiction. Hence $\mathbb{Q}H$ has the desired components.

Suppose now that G is nilpotent. □

2.3. **Groups with low character degrees.** This section and the next aim to give a precise description of the groups satisfying $(\mathrm{M}_{\mathrm{exc}})$. By Theorem 2.1 we should first consider the more general class of groups with the character degrees of the irreducible complex characters all divisors of 4. We denote by $\mathrm{cd}(G)$ the set of character degrees of irreducible complex representations of G. In this section we focus on the groups G with $\mathrm{cd}(G) = \{1, 4\}$.

More generally, whenever $\mathrm{cd}(G) \subseteq \{1, p^j, p^k\}$ then G is solvable of derived length at most 3 [9, Theorem 12.15]. Furthermore, $(\chi(1), q) = 1$ for any prime q different from p and any $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$. This allows to apply the Ito-Michler theorem [9, Corollary 12.34] which in this case yields that

$$G \cong A \rtimes Q, \tag{2.2}$$

where A is an abelian p'-subgroup and Q a Sylow p-subgroup of G. More precisely, A is the direct product of all Sylow q-subgroups of G, for all $q \neq p$, which are normal and abelian under the above assumption on $\mathrm{cd}(G)$.

Using character theory and [8], the following results will give further restrictions on the decomposition (eq. (2.2)) in the case that $|\mathrm{cd}(G)| = 2$.

Proposition 2.9. *Let G be a non-nilpotent group such that $\mathrm{cd}(G) = \{1, p^k\}$, with $k > 1$. Then the Fitting subgroup $F(G)$ of G is the unique maximal abelian subgroup of index p^k. Furthermore, all Sylow subgroups of G are abelian, and G decomposes as a semidirect product*

$$G \cong N \rtimes C,$$

with N an abelian subgroup of $F(G)$ and C a 2-generated p-group $\langle x, y\rangle$, with x acting with order p^k, and y central in G. Conversely, a group G as above with $|\mathrm{cd}(G)| = 2$ must have $\mathrm{cd}(G) = \{1, p^k\}$.

Remark 2.10. The proof of Proposition 2.9 will also yield extra restrictions on the structure of G. For example $F(G) = C_G(A) \cong Z(G) \times [G, G]$ by (2.3) and (2.4). Additionally, denoting $C = \langle x, y\rangle$ and using [8, Theorem 3.1.(iii)], one can prove that the action of x^i, for $1 \leq i \leq p^k$, on $[G, G]$ is free.

[10] Add justification...

Proof of Proposition 2.9. As noticed earlier G has the form (2.2) with A and Q as described there. If Q is non-abelian, then [9, Exercise 12.6] implies that G is nilpotent, which is in contradiction with the hypothesis. Hence Q is abelian.

Now we show that the Fitting subgroup is the centraliser of A, i.e. $F(G) = C_G(A)$. By[11] [8, Theorem 2.2 (ii)], $C_G(A)$ is a normal abelian subgroup of G. As $F(G)$ is the unique maximal normal nilpotent subgroup, the inclusion $C_G(A) \subseteq F(G)$ follows. But by [19, Lemma 1.2 (a)], there is some character χ of G such that $[G : F(G)] = \chi(1)$. Since G is non-nilpotent by assumption, we obtain that

$$p^k = [G : F(G)] \leq [G : C_G(A)] = p^k,$$

where the last equality follows from [8, Theorem 2.2 (ii)]. Thus in particular

$$C_G(A) = F(G). \tag{2.3}$$

Next, since Q is abelian one can apply [8, Theorem 3.1 (ii)], implying that $G/F(G)$ is cyclic of order p^k. As G is of the form (2.2), we can choose an $x \in Q$ such that $\bar{x} := xF(G)$ generates $G/F(G)$. As $\bar{x}$ has order p^k, it follows that $F(G) \cap \langle x \rangle = \langle x^{p^k} \rangle$.

Let $\langle y \rangle$ be the cyclic subgroup of $F(G)$ containing x^{p^k} and which is maximal amongst the cyclic subgroups of $Q \cap F(G)$ for this property. As Q is abelian, there exists a $M \leqslant Q$ such that $Q \cap F(G) \cong M \times \langle y \rangle$. Note that since $F(G)$ is abelian, eq. (2.2) implies that

$$F(G) = A \times (Q \cap F(G)),$$

and it follows that

$$F(G) = N \times \langle y \rangle, \text{ with } N = A \times M.$$

We conclude the proof by showing that

$$G \cong N \rtimes \langle x, y \rangle.$$

Firstly, it is clear that $N \cap \langle x, y \rangle = \{1\}$, since if $t \in N \cap \langle x, y \rangle$, then $t = x^r y^s$ for some $r, s \in \mathbb{Z}$. But then $ty^{-s} \in F(G)$, which since $F(G) \cap \langle x \rangle = \langle x^{p^k} \rangle$, implies that $ty^{-s} = (x^{p^k})^\ell$ for some $\ell \in \mathbb{Z}$. But $\langle x^{p^k} \rangle \leqslant \langle y \rangle$, and in particular there exists an $s' \in Z$ with $t = y^{s'}$. But $N \cap \langle y \rangle = \{1\}$ by construction, and hence $t = 1$.

Additionally, $N \,.\, \langle x, y \rangle = G$, because

$$|N||\langle x, y \rangle| = |N||\langle y \rangle| \frac{|\langle x \rangle|}{|\langle x^{p^k} \rangle|} = |F(G)|[G : F(G)].$$

Finally, we show that N is normal in G. Combining [8, Theorem 3.1 (iii)] with eq. (2.3), [19, Lemma 1.6 (d)] now implies that

$$F(G) = C_G(A) \cong \mathcal{Z}(G) \times G'. \tag{2.4}$$

Since $N = A \times M$, and A is normal in G by construction, it suffices to show that M is normal. We show that in fact $\langle y \rangle \times M$ is even central in G. Indeed, if $z \in \langle y \rangle \times M \leqslant F(G)$, it may be written as $z = z_1 z_2$ for unique $z_1 \in \mathcal{Z}(G)$ and $z_2 \in G'$ by the above. Furthermore, since Q is an abelian Sylow p-subgroup and so $G' \leqslant A$, one has that $(o(z_2), p) = 1$. But by definition z has order a power of p, and it follows that

$$\langle z \rangle = \langle z^{o(z_2)} \rangle \subseteq \mathcal{Z}(G),$$

as claimed. Finally, the action of x on N is of order p^k, since x^{p^k} is the smallest non-trivial power of x which commutes with N, because $F(G) = \langle N, x^{p^k} \rangle$ is the maximal normal abelian subgroup.

For the converse, as $[G : F(G)] = p^k$ and $F(G)$ is a normal abelian subgroup, Ito's theorem [9, Theorem 6.15] yields that $\chi(1)$ divides p^k for all irreducible characters χ. Note also that $G' \leq N \leq F(G)$, hence by [7, Lemma 1] there exists some $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ with $\chi(1) = [G : F(G)] = p^k$. The assumption $|\operatorname{cd}(G)| = 2$ concludes the proof. ☐

[11] The fact that the Sylow p-subgroup Q is abelian yields that G is a $\{p\}$-character group as in [8, Definition 2.1.]

Example 2.11. Consider the group $G := C_5 \rtimes C_8 = \langle a, b \mid a^5, b^8, a^b = a^3 \rangle$. It is easily shown that $\mathrm{cd}(G) = \{1, 4\}$ and $F(G) = C_G(\langle a \rangle) = \langle a, b^4 \rangle$. Thus, in the notation of Proposition 2.9, $N = \langle a \rangle$ and $C = \langle b \rangle$. We see that in this example one can indeed not find a complement for $F(G)$ itself. Thus the splitting in Proposition 2.9 is the finest possible in general.

Proposition 2.12. *Let G be a non-nilpotent finite group such that $\mathrm{cd}(G) \subseteq \{1, p^i, p^k\}$ with $i < k$. Using the notations from (2.2), if Q is abelian then the following hold:*

(1) $C_G(A) = F(G) = \mathcal{Z}(G) \times G'$,
(2) $F(G)$ is the unique maximal abelian subgroup, of index p^k,
(3) $G/C_G(A)$ is either cyclic of order p^k or isomorphic to $C_{p^i} \times C_{p^{k-i}}$.
(4) Something on decomposition of $C_G(A)$ and how $G/C_G(A)$ acts on it;

Proof. As explained at the beginning of the section, the group G has the form (2.2) with A and Q as described there, and G is solvable. Since Q is assumed abelian, in particular G is solvable of derived length at most 2. It follows from the work of Taunt, [6, VI, Satz 14.7 b)] that $F(G) = \mathcal{Z}(G) \times G'$. Additionally, $C_G(A)$ and $G/C_G(A)$ are abelian and in particular it follows from [6, VI, Satz 14.7 a)] that $C_G(A) = \mathcal{Z}(G) \times (C_G(A) \cap G') = \mathcal{Z}(G) \times G'$. It now also immediately follows that $F(G)$ is the unique maximal *abelian* subgroup. By the fact that G is non-nilpotent by assumption, we obtain as in the proof of Proposition 2.12 that $[G : F(G)] \in \{p^i, p^k\}$, and $[G : C_G(A)] = p^k$. Hence $F(G)$ is of index p^k. Now, by [8, Theorem 2.2], $C_G(A)$ is either isomorphic to C_{p^k} or to $C_{p^i} \times C_{p^{k-i}}$. □

Next consider the nilpotent case. Then the decomposition in (2.2) is a direct product $G \cong A \times Q$. A precise classification in the nilpotent case seems hard. Nevertheless note that $\{1, 4\} = \mathrm{cd}(G) = \mathrm{cd}(Q)$. Now applying [8, Theorem 3.10 & Lemma 5.4] yields the following.

Lemma 2.13. *Let G be a nilpotent group with $\mathrm{cd}(G) = \{1, 4\}$, then $G \cong A \times Q$ with A an odd abelian group and Q a 2-group satisfying the following:*

(1) Q has nilpotency class 2,
(2) $[Q, Q]$ and $Q/\mathcal{Z}(Q)$ are elementary abelian 2-groups.

For $\mathrm{cd}(G) = \{1, 2, 4\}$ not necessarily true, see [16,8]. Maybe still $G'^2 \subseteq \mathcal{Z}(G)$?

2.4. **Characterisation of the groups.** In this section we will give a complete characterisation of groups satisfying property $(\mathrm{M}_{\mathrm{exc}})$. Recall from (2.2) that $G \cong A \rtimes Q$ with A an odd abelian group and Q a 2-group. Denote by $\varphi : Q \to \mathrm{Aut}(A)$ the action of Q on A.

In Theorem 2.6 it was proven that such G contain a index two subgroup H such that all non-division simple components of $\mathbb{Q}H$ are of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ with $d \in \mathbb{Z}_{\geq 0}$. Such groups have already been addressed in [15]. Therefore we will focus on the new cases, i.e. when $\mathrm{M}_2\left(\left(\frac{-a,-b}{\mathbb{Q}}\right)\right) \in \mathcal{C}(\mathbb{Q}G)$ for some $a, b > 0$. For groups having $(\mathrm{M}_{\mathrm{exc}})$ this is equivalent to the fact that 4 is a character degree of G.

Maybe put that equivalence centrally...

Non-nilpotent groups with $(\mathrm{M}_{\mathrm{exc}})$.

Theorem 2.14. *Let G be a non-nilpotent group with $(\mathrm{M}_{\mathrm{exc}})$. With notations as above the following holds:*

(1) A is an elementary abelian 3-group,
(2) $[G : F(G)] = [Q : \mathrm{Ker}(\varphi)] \mid 4$ **aantonen dat het 2 is en als gevolg dat fitting non-abelian**
(3) something with action by inversion modulo some central part?

If furthermore $4 \in \mathrm{cd}(G)$, then $\mathcal{Z}(G)$ is an elementary abelian 2-group.

Ik weet niet hoe aan te tonen dat $\mathcal{Z}(Q)$ een elementair abelse 2-groep is.

Proof. We show the statement by induction on the cardinality of Q. Indeed, suppose that when $|Q| = 2$, A is an elementary abelian 3-group. Now suppose $G = A \rtimes Q$ has $(\mathrm{M}_{\mathrm{exc}})$ for some 2-group Q. Let $P \leqslant Q$ of index 2, and define the subgroup $H = A \rtimes P$. Then H is a normal subgroup of G, since it is of index 2. In particular, H has $(\mathrm{M}_{\mathrm{exc}})$ by Lemma 2.8. If H is non-nilpotent, it follows by the induction hypothesis that A is an elementary abelian 3-group. If H is nilpotent, then $H = A \times P$, and in particular the kernel $\mathrm{Ker}(\varphi)$ of the action of Q on A contains P. Since G is non-nilpotent by assumption and P has index 2 in Q, $\mathrm{Ker}(\varphi) = P$. Then $G/\mathrm{Ker}(\varphi) \cong A \rtimes (Q/\mathrm{Ker}(\varphi)) \cong A \rtimes C_2$. By Lemma 2.3, $G/\mathrm{Ker}(\varphi)$ has ($\mathrm{M_{exc}}$), and it follows by assumption that A is an elementary abelian 3-group.

We proceed to show that when $Q \cong C_2$, A is indeed an elementary abelian 3-group. Since A is abelian, in particular it is given by $A = \prod_{q\in\pi(A)} A_q$, with A_q the Sylow q-subgroup of G, which is a characteristic subgroup of G for every $q \in \pi(A)$. We can choose $p \in \pi(A)$ such that G/R_p, with $R_p := \prod_{q\in\pi(A)\setminus\{p\}} A_q$, is non-nilpotent. Indeed, if $G/R_p \cong A_p \rtimes C_2$ were nilpotent for every $p \in \pi(A)$, it would follow that C_2 acts trivially on every Sylow q-subgroup of G, and in particular $G \cong A \times C_2$, a contradiction with the non-nilpotency of G. Now from [8, Lemma 1.2] it follows that $A_p \cong (A_p \cap \mathcal{Z}(A_p \rtimes C_2)) \times B_p$ for some characteristic subgroup $B_p \leqslant A_p \rtimes C_2$. Additionally, B_p is non-trivial, since otherwise $A_p \leqslant \mathcal{Z}(A_p \rtimes C_2)$, a contradiction with the assumption on p. Now,

$$B_p \rtimes C_2 \cong G/\left(R_p \cdot (\mathcal{Z}(A_p \rtimes C_2) \cap A_p)\right),$$

and in particular $B_p \rtimes C_2$ has ($\mathrm{M_{exc}}$) by Lemma 2.3. Let $x \in B_p$ have maximal order, say $o(x) = p^m$. Then there is a subgroup $K \leqslant B_p$ such that $B_p \cong \langle x\rangle \times K$ (since $B_p \leqslant A_p$ is abelian). We claim that (B_p, K) is a strong Shoda pair for $B_p \rtimes C_2$. Indeed, to find an $H \leqslant B_p \rtimes C_2$ such that (H, K) is a SSP for $B_p \rtimes C_2$, by [14, Theorem 3.5.12] it suffices to show that B_p is a maximal element in the set

$$S := \{D \leqslant B_p \rtimes C_2 \mid B_p \leqslant D \text{ and } D' \leqslant K \leq D\}.$$

Since $[B_p \rtimes C_2 : B_p] = 2$, it follows that $D \in S$ can only occur if $D = B_p$ or $D = B_p \rtimes C_2$. We claim that $(B_p \rtimes C_2)' \nleqslant K$, and hence $D \in S$ if and only if $D = B_p$. Indeed, for any $\bar{a} \in B_p$, let $a \in A_p$ such that $a \mapsto \bar{a}$ under the quotient map $\varpi\colon G \to B_p \rtimes C_2$. Let $C_2 = \langle y\rangle$. Then aa^y commutes with y: $(aa^y)^y = a^y a^{y^2} = a^y a = aa^y$, where the last equality follows since $a^y, a \in A_p$ by definition, and A_p is abelian. Hence $aa^y \in \mathcal{Z}(A_p \rtimes C_2)$. It follows by definition of B_p that $\bar{a}\bar{a}^y = \varpi(aa^y) = 1$. In particular, $\bar{a}^y = \bar{a}^{-1}$ in $B_p \rtimes C_2$. We conclude that $[\bar{a}, y] = \bar{a}^{-1}\bar{a}^y = \bar{a}^{-2} \in \langle\bar{a}\rangle$. In particular, for $\bar{a} = x$, it follows that $[x, y] \notin K$, and more generally we obtain that $y \in N_{B_p\rtimes C_2}(B_p)$.

Combining the latter fact with [10, Lemma 3.4], it follows that

$$\begin{aligned}\dim_{\mathbb{Q}}\left(\mathbb{Q}[B_p \rtimes C_2]e(B_p \rtimes C_2, B_p, K)\right) &= [B_p \rtimes C_2 : B_p] \cdot [B_p \rtimes C_2 : N_{B_p\rtimes C_2}(K)] \cdot \phi\left([B_p : K]\right)\\ &= 2 \cdot 1 \cdot p^{m-1}(p-1).\end{aligned}$$

However, since $B_p \rtimes C_2$ has ($\mathrm{M_{exc}}$) by construction, the above $\mathbb{Q}$-dimension is an element of $\{4, 8, 16\}$. It follows that $p \in \{3, 5\}$ and $m = 1$. In particular, by definition of m, A_p is an elementary abelian p-group. Suppose $p = 5$. Then since y acts on B_p by inversion, it follows from [14, Theorem 3.5.5 (4) and Remark 3.5.6] that

$$\mathbb{Q}(\zeta_5 + \zeta_5^{-1}) \subseteq \mathcal{Z}(\mathbb{Q}[B_p \rtimes C_2]e(B_p \rtimes C_2, B_p, K)).$$

However, $|\mathbb{Q}(\zeta_5 + \zeta_5^{-1}) : \mathbb{Q}| = 2$, and $\mathbb{Q}(\zeta_5 + \zeta_5^{-1})$ is a totally real field. But by Theorem 2.1, if $B_p \rtimes C_2$ has ($\mathrm{M_{exc}}$), the centres of its 1×1 components intersected with $\mathbb{R}$ should equal $\mathbb{Q}$. Since $B_p \rtimes C_2$ has ($\mathrm{M_{exc}}$) by construction, we obtain a contradiction. Hence A_p is an elementary abelian 3-group.

In summary, we have proven that G decomposes as $G \cong A_{3'} \times (A_3 \rtimes Q)$ with A_3 an elementary abelian subgroup. **add why there can't be $p \geq 5$ in the center (or certainly not as direct factor), maybe by refering to previous result. Hence $A = A_3$.**

Next we prove statement (2). By Theorem 2.6 G has a subgroup H of index 2 such that all its non-division components are of the form $\mathrm{M}_n(\mathbb{Q}(\sqrt{-d}))$. In particular, $\mathrm{cd}(H) \subseteq \{1, 2\}$ by Theorem 2.1. Hence, from [2, Theorem 3] it follows that either H has an abelian subgroup $B \leqslant H$ of index 2, or $H/\mathcal{Z}(H) \cong C_2 \times C_2 \times C_2$. In the latter case, $H' \subseteq \mathcal{Z}(H)$, and in particular H is nilpotent of class 2. Thus independent of the case, H contains a nilpotent subgroup C of index at most 2 in H. Hence there exists a nilpotent subgroup $C \leqslant G$

such that $[G:C] \mid 4$. Now it follows that $A \leqslant C$, since $|A|$ is of odd order. In particular, $C = A \times (C \cap Q) \leqslant A \times \mathrm{Ker}(\varphi)$ since C is nilpotent. But then $[Q : \mathrm{Ker}(\varphi)] \mid 4$, since

$$[G:C] = [G/A : C/A] = [Q : (C \cap Q)] = [Q : \mathrm{Ker}(\varphi)][\mathrm{Ker}(\varphi) : (C \cap Q)],$$

and $[G : C] \mid 4$. The fact that $[Q : \mathrm{Ker}(\varphi)] = [G : F(G)]$ follows immediately since $A \leq F(G)$. □

Nilpotent groups with ($\mathrm{M}_{\mathrm{exc}}$)*.*

Theorem 2.15. *Let G be a nilpotent group satisfying* ($\mathrm{M}_{\mathrm{exc}}$) *and with* $4 \in \mathrm{cd}(G)$*. Then*

(1) G is a 2-group
(2) $G' \cap \mathcal{Z}(G) = ...$ (or with the square of commutator
(3) G has nilpotency class at most 3
(4) something with exponent

Theorem 2.16. *Let G be a finite group satisfying* ($\mathrm{M}_{\mathrm{exc}}$)*. If* $\mathrm{cd}(G) = \{1, 4\}$*, then ...*

3. The block Virtual Structure Problem

Recall definition of SL_1

Question 3.1 (block Virtual Structure problem)**.** Let $\mathcal{P}$ be a property. Classify the group algebras FG such that $\mathrm{SL}_1(FGe)$ has property $\mathcal{P}$ for every $e \in \mathrm{PCI}(FG)$.

3.1. **On a generalization of Kleinert - Del Rio.** Let $\mathcal{P}$ be a group theoretical property such that

Think which props exactly to put

- $\mathcal{P}$ implies *not* FA or not SCP
- SL_1 of all exceptional 2×2 components satisfy it
- it is a property of commensurability classes.

Example 3.2. **Is $\mathcal{G}_{am}$ er zo eentje? Als niet, welk wel?** Al die uit de paper van Zalesski-Del rio

By $\prod \mathcal{P}$ we mean that the group is a direct product of groups satisfying $\mathcal{P}$ *and* an abelian group. Combining the methods in the proof of [11, Theorem 8.2] **nummer updaten** and [15, Theorem 2.1.] with the results of [16] we obtain the following.

Proposition 3.3. *Let A be a finite dimensional semisimple F-algebra with F a number field and $\mathcal{O}$ an order in A. If $\mathcal{U}(\mathcal{O})$ is virtually-$\prod \mathcal{P}$, then for every $e \in \mathrm{PCI}(A)$:*

(1) $\mathrm{SL}_1(\mathcal{O}e)$ *is either virtually-$\mathbb{Z}$ or virtually-$\mathcal{P}$*
(2) The degree of Ae, as CSA, is at most 4

Remark 3.4. Proposition 3.3 shows that for a property constant on commensurability and direct products the classical Virtual structure problem is equivalent to the block VSP (Question 3.1).

Begin proof Proposition 3.3. Part 1: Beetje zoals die claim bij onze vorige paper maar nu opsplitsen in de abelse deel en de amalgam deel en dan argumenteren dat central en de SL_1 maar eindig snijden en dus de product van amalgams van finite index nog in de SL_1. Op dat punt zoals ons bewijs.

Part 2: This is more generally the case for a property which imply not FA, cf. work of Kleinert-Del Rio where they deduce this by considering the associated semisimple Lie group and do rank computations. □

3.2. **Groups of virtual cohomological dimension 4.** Let Γ be a discrete group. Then the *cohomological dimension* of Γ over the ring R is

$$\mathrm{hdim}_R \Gamma := \min\{n \mid H^k(G,M) = 0 \text{ for all } k > n \text{ and } M \in \mathrm{mod}(RG)\}.$$

If no such n exists one says that $\mathrm{hdim}_R \Gamma = \infty$. A usual obstruction to have a finite cohomological dimension is torsion in Γ. However, if Γ has a torsion-free subgroup of finite index (e.g. Γ is linear), then each of such finite index subgroups has the same cohomological dimension. Hence

$$\mathrm{vcd}(\Gamma) := \{\mathrm{hdim}_{\mathbb{Z}} \Gamma' \mid [\Gamma : \Gamma'] < \infty \text{ and } \Gamma' \text{ torsion-free}\}.$$

The finite dimensional simple F-algebras with F a number field and such that $\mathrm{vcd}(\mathrm{SL}_1(A)) \leq 2$ have been classified in [15, Proposition 3.3]. We extend this result to virtual cohomological dimension 4.

Proposition 3.5. *Let A be a finite dimensional simple F-algebra with F a number field. If $\mathrm{vcd}(\mathrm{SL}_1(A)) = 4$, then A is of one of the following forms:*

(1) $\mathrm{M}_2\left(\left(\frac{-a,-b}{\mathbb{Q}}\right)\right)$ *with* $a, b \in \mathbb{N}_0$,
(2) $\mathrm{M}_2(F)$, *with F a cubic field with precisely one real embedding and one pair of complex embeddings,*
(3) $\left(\frac{-a,-b}{F}\right)$ *with F totally real and non-ramified at exactly two real places.*

Proof. **It should be** $\mathrm{SL}_1(\mathcal{O}$**? (for which topology?)** Let $A = M_n(D)$, $F = \mathcal{Z}(D)$ for an integer $n \geq 1$ and D a division ring of degree d. We make use of the following formula, as stated in [15, Eq. (1)].

$$\mathrm{vcd}(\mathrm{SL}_1(A)) = r_1 \frac{(nd-2)(nd+1)}{2} + r_2 \frac{(nd+2)(nd-1)}{2} + s(n^2d^2-1) - n + 1, \tag{3.1}$$

where s is the number of pairs of non-real complex embeddings of F, r_1 is the number of real embeddings of F at which A is ramified, and r_2 the number of real embeddings of F at which A is not ramified. We may assume that $nd > 1$, since when $nd = 1$, A is a field, which only happens when $\mathrm{vcd}(\mathrm{SL}_1(A)) = 0$ by [15, Proposition 3.3]. Note that for any choice of $nd \geq 2$, the first two terms of eq. (3.1) are non-negative. Additionally, when d is odd, it immediately follows that $r_1 = 0$.

Ik ben totaal niet zelfzeker over deze terminologie dus dit moet nog bekeken worden. Voorlopig heb ik het gewoon overgenomen uit de free by free paper maar ik vind het raar

Suppose $s \geq 2$. Then for any choice of n or d such that $nd = 2$,

$$\mathrm{vcd}(\mathrm{SL}_1(A)) \geq s(n^2d^2-1) - n + 1 > 4,$$

and this expression is strictly increasing in both n and d. Hence $\mathrm{vcd}(\mathrm{SL}_1(A)) > 4$ for any $s \geq 2$, and in particular F has at most one pair of complex embeddings.

Suppose $s = 1$. Then

$$\mathrm{vcd}(\mathrm{SL}_1(A)) \geq s(n^2d^2-1) - n + 1 = n^2d^2 - n > 4,$$

when $n \geq 3$. Hence n is at most 2 in this case. Suppose first $n = 1$. Then

$$\mathrm{vcd}(\mathrm{SL}_1(A)) \geq d^2 - 1 > 4 \text{ when } d > 2.$$

Thus $d = 2$ because $nd \geq 2$ by assumption. Then we find

$$\mathrm{vcd}(\mathrm{SL}_1(A)) = r_2 \frac{4.1}{2} + 4 - 1 = 4 \iff r_2 = -\frac{1}{2},$$

which is a contradiction since $r_2 \in \mathbb{N}$. Hence we cannot have $n = 1$. The only other option is $n = 2$. Then $n^2d^2 - n = 4d^2 - 2 \leq 4$ if and only if $d = 1$, in which case

$$\mathrm{vcd}(\mathrm{SL}_1(A)) = r_2 \frac{4.1}{2} + 2 = 4 \iff r_2 = 1.$$

In this case we find that $A = M_2(F)$ with F a cubic number field with precisely one real embedding and one pair of complex embeddings.

Suppose now $s = 0$. We examine the cases $r_2 = 0$ first. Suppose $r_2 = 0$. Then $r_1 \geq 1$ and

$$\mathrm{vcd}(\mathrm{SL}_1(A)) = r_1 \frac{(nd-2)(nd+1)}{2} - n + 1,$$

and hence $\mathrm{vcd}(\mathrm{SL}_1(A)) = 4$ necessarily implies that $nd \geq 3$. Suppose first that $n = 1$. Then

$$\mathrm{vcd}(\mathrm{SL}_1(A)) = r_1 \frac{(d-2)(d+1)}{2} = 4 \iff r_1 \leq 2, \text{ since } d \geq 3.$$

If $r_1 = 2$, then $\mathrm{vcd}(\mathrm{SL}_1(A)) = (d-2)(d+1) = 4$ implies that $d = 3$, but we have already remarked that $r_1 = 0$ when d is odd, a contradiction. If $r_1 = 1$, then $(d-2)(d+1) = 8$, which implies that $d = \frac{2 \pm \sqrt{32}}{2}$, which is not an integer, a contradiction. We conclude that if $s = 0$ and $r_2 = 0$, then $n > 1$. Suppose $n = 2$. Then necessarily $d \geq 2$ since $nd \geq 3$. It follows that

$$\mathrm{vcd}(\mathrm{SL}_1(A)) = \frac{r_1}{2}(2d-2)(2d+1) - 1 \geq 4,$$

with equality if and only if $r_1 = 1$ and $d = 2$. We find that in this case, $A = M_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ for some positive integers a, b. Now the expression $\frac{r_1}{2}(nd-2)(nd+1) - n + 1$ is strictly increasing in n if and only if $n > \frac{d-1}{r_1 d^2}$, which in the case at hand is satisfied since $d \geq 1$ and $r_1 \geq 1$ by assumption. It follows that $\mathrm{vcd}(\mathrm{SL}_1(A)) > 4$ whenever $n \geq 3$.

Still under the assumption that $s = 0$, we now turn our attention to the case $r_2 \neq 0$, meaning $r_2 \geq 1$. Then

$$\mathrm{vcd}(\mathrm{SL}_1(A)) = \frac{r_2}{2}(nd+2)(nd-1) + \frac{r_1}{2}(nd-2)(nd+1) - n + 1.$$

If $r_1 \geq 1$, then d is even, meaning that $nd \geq 4$ when $n \geq 2$. But when $n = 2$, then

$$\frac{r_2}{2}(2d+2)(2d-1) + \frac{r_1}{2}(2d-2)(2d+1) - 1 \geq 9r_2 + 5r_1 - 1 > 4,$$

and again the expression above for $\mathrm{vcd}(\mathrm{SL}_1(A))$ is strictly increasing in n. If $r_1 = 0$, then $\mathrm{vcd}(\mathrm{SL}_1(A)) = 4$ if and only if $\frac{r_2}{2}(nd+2)(nd-1) - n = 3$. This expression is strictly increasing in n if and only if $n > \frac{1-d}{r_2 d^2}$, which is always satisfied by assumption on r_2 and d. But when $n \geq 2$ and $d \geq 2$, meaning in particular that $nd \geq 4$, one finds that

$$\frac{r_2}{2}(nd+2)(nd-1) - n \geq 7,$$

implying that only the cases $n = 1$, and $(n, d) = (2, 1)$ need to be investigated. If $n = 2$ and $d = 1$, then

$$\mathrm{vcd}(\mathrm{SL}_1(A)) = \frac{r_2}{2}(nd+2)(nd-1) - n + 1 = 2r_2 - 1,$$

which equals 4 if and only if $r_2 = \frac{5}{2}$, a contradiction.

In particular, for any value of r_1 only the case $n = 1$ remains. Assuming $n = 1$, we have in particular that $d \geq 2$, and

$$\mathrm{vcd}(\mathrm{SL}_1(A)) = \frac{r_2}{2}(d+2)(d-1) + \frac{r_1}{2}(d-2)(d+1) \geq \frac{r_2}{2}(d+2)(d-1) \geq 4,$$

where the last inequality becomes an equality if and only if $d = 2 = r_2$. In that case one also finds that $\frac{r_1}{2}(d-2)(d+1) = 0$, and hence $\mathrm{vcd}(\mathrm{SL}_1(A)) = 4$ if $s = 0$, $r_2 = d = 2$ and r_1 takes any arbitrary integer value. We obtain that $A = \left(\frac{-a,-b}{F}\right)$ for F a totally real number field which is ramified at precisely two places. If $d \geq 3$, it now immediately follows that $\mathrm{vcd}(\mathrm{SL}_1(A)) \geq 9r_2 + 4r_1 > 4$, and this concludes our analysis.

Die terminologie is hoe Doryan en ik het hebben opgeschreven na onze berekening, maar komt niet overeen met de definitie van r_1 en r_2 zoals in het begin van het bewijs, en die definitie is overgenomen van de free by free paper

□

Remark 3.6. With a similar proof one can verify that $\mathrm{vcd}(\mathrm{SL}_1(A)) = 3$ if and only if A is isomorphic to one of the following simple algebras:

- $\mathrm{M}_3(\mathbb{Q})$,
- $\mathrm{M}_2(\mathbb{Q}(\sqrt{d}))$ with $d \in \mathbb{N}$ square-free,

- $\left(\frac{-a,-b}{F}\right)$ such that F has one pair of (non-real) complex embeddings and is ramified at all real places.

Concerning $\mathrm{vcd}(\mathrm{SL}_1(A)) \leq 2$, [15, Proposition 3.3] tells that

- $\mathrm{vcd}(\mathrm{SL}_1(A)) = 0$ if and only if A is a field or a totally definite quaternion algebra,
- $\mathrm{vcd}(\mathrm{SL}_1(A)) = 1$ if and only if $A \cong \mathrm{M}_2(\mathbb{Q})$
- $\mathrm{vcd}(\mathrm{SL}_1(A)) = 2$ if and only if $A \cong \mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ or a quaternion algebra with a totally real centre and which is non-ramified at exactly one infinite place.

We now have all ingredients for our following main theorem.

Theorem 3.7. need to say no exceptional type I? *Let G be a finite group. Then the following are equivalent:*

(1) $(\mathrm{M}_{\mathrm{exc}})$
(2) $\mathrm{vcd}(\mathrm{SL}_1(\mathbb{Q}Ge))$ *is* 0 *or a divisor of* 4 *for each* $e \in \mathrm{PCI}(\mathbb{Q}G)$

The proof of Theorem 3.7 will follow quickly out of the results obtained in earlier sections together with following fact which is of independent interest.

TO do: adapt following proof and statement to arbitrary KG with K a number field.

Lemma 3.8. *Let G be a finite group. Suppose that the simple algebra $\mathrm{M}_2(F)$ is a quotient of $\mathbb{Q}G$, with F a cubic extension of $\mathbb{Q}$, say $\mathrm{M}_2(F) \cong \mathbb{Q}Ge$ with $e \in \mathrm{PCI}(\mathbb{Q}G)$. Then F is totally real and $\pi(Ge) \subseteq \{2,3,7\}$.*

Example 3.9. **Zet een voorbeeld waar er ee totlly real is.**

Proof. Let $\lambda \in F$ be a torsion element, say of order n. Then it is a primitive n^{th} root of unity, denoted ζ_n, and

$$3 = |F:\mathbb{Q}| = |F:\mathbb{Q}(\zeta_n)||\mathbb{Q}(\zeta_n):\mathbb{Q}| \geq \phi(n). \tag{3.2}$$

It follows that $n \in \{1,2,3,4,6\}$.

Let $g \in G$. By fixing a $\mathbb{Q}$-basis of F as a 3-dimensional $\mathbb{Q}$-space, one can realise g as a 6-by-6 matrix over $\mathbb{Q}$. We denote by $\chi_{F,g}$ and $\chi_{\mathbb{Q},g}$ respectively the characteristic polynomials of g over F and over $\mathbb{Q}$. Similarly, we write $\mu_{F,g}$ and $\mu_{\mathbb{Q},g}$ for the minimal polynomials of g over respectively F and $\mathbb{Q}$. By definition, any minimal polynomial $\mu_{\mathbb{Q},g}$ has degree at most 6, and any $\mu_{F,g}$ has degree at most 2. Remark that $\mu_{F,g}$ has degree 1 if and only if g is a scalar matrix over F.

From [17, Page 147], it follows that for any $g \in G$ of prime power order p^k, the $p^{k\text{th}}$ cyclotomic polynomial Φ_{p^k} equals $\mu_{\mathbb{Q},g}$. In particular, $\mathbb{Q}(\zeta_{p^k}) \cong \frac{\mathbb{Q}[X]}{(\mu_{\mathbb{Q},g})}$. The latter also holds over F, when $p^k > 4$. Indeed, $\mu_{F,g}$ is given as the unique monic polynomial generating the ideal $I_g := \{P \in F[X] \mid P(g) = 0\}$, and the minimal polynomial of ζ_{p^k} over F is given by the unique monic polynomial generating the ideal $I_{\zeta_{p^k}} := \{Q \in F[X] \mid Q(\zeta_{p^k}) = 0\}$. We claim that $I_g = I_{\zeta_{p^k}}$. Indeed, by [17, Page 146], there is some matrix $B \in \mathrm{GL}_2(\mathbb{C})$ such that $BgB^{-1} = \mathrm{diag}(\lambda_1, \lambda_2)$, with each λ_i a $p^{k\text{th}}$ root of unity, amongst which at least one primitive (otherwise BgB^{-1} and hence g would have order strictly smaller than p^k). Without loss of generality, assume $\lambda_1 = \zeta_{p^k}$. Remark that since conjugation by an invertible matrix induces an algebra automorphism of $\mathrm{M}_2(\mathbb{C})$, it follows that $P \in I_g$ if and only if $P(BgB^{-1}) = 0$. In particular, $P \in I_g$ if and only if $P(\zeta_{p^k}) = 0 = P(\lambda_2)$. We conclude that $I_g \subseteq I_{\zeta_{p^k}}$. But since $\deg(\mu_{F,g}) = 2$, and $p^k > 4$ (meaning that $\zeta_{p^k} \notin F$), it follows that I_g is a maximal ideal of $F[X]$, and hence $I_g = I_{\zeta_{p^k}}$. In particular, $F(\zeta_{p^k}) \cong \frac{F[X]}{(\mu_{F,g})}$.

Since Φ_p divides $\mu_{\mathbb{Q},g}$ when $g \in G$ is an element of prime order p, and the degree of $\mu_{\mathbb{Q},g}$ is at most 6 as remarked earlier, it follows that $p \in \{2,3,5,7\}$, and in particular $\pi(G) \subseteq \{2,3,5,7\}$. Suppose $g \in G$ has order $p \in \{5,7\}$. Remark that $\deg(\chi_{F,g}) = 2$, since otherwise g would be a scalar matrix with a p^{th} root of unity on the diagonal, which is a contradiction with the description of torsion elements in F as given in eq. (3.2). Over $F(\zeta_p)$,

$\chi_{F,g}$ splits as $(X-\zeta_p^i)(X-\zeta_p^k)$ for some $0 \leq i \neq k \leq p-1$. Now $\zeta_p^{i+k} \neq 1$ would imply that F has a p^{th} root of unity, a contradiction by eq. (3.2). Thus, $k=-i$, and the degree 1 coefficient in $\chi_{F,g}$ is equal to $-(\zeta_p^i+\zeta_p^{-i})$. In particular $\mathbb{Q}(\zeta_p^i+\zeta_p^{-i}) \subseteq F$. If $p=5$, then since $|\mathbb{Q}(\zeta_5+\zeta_5^{-1}):\mathbb{Q}|=2$ does not divide $|F:\mathbb{Q}|=3$, we obtain a contradiction. When $p=7$, $|\mathbb{Q}(\zeta_7+\zeta_7^{-1}):\mathbb{Q}|=3$. It follows that $F=\mathbb{Q}(\zeta_7+\zeta_7^{-1})$.

Let now $\pi(G) \subseteq \{2,3\}$. We bound the exponent of G. Let $g \in G$, say of order 2^i3^j for some non-negative integers i,j. If $i=0$ or $j=0$, then $o(g)=p^n$ since Φ_{p^n} divides $\mu_{\mathbb{Q},g}$ for each $p \in \{2,3\}$ and $\deg(\mu_{\mathbb{Q},g}) \leq 6$, it follows that $o(g) \in \{p,p^2\}$. If $i \neq 0 \neq j$, then considering a realisation of g as an element in $\mathrm{GL}_6(\mathbb{Q})$, [17, Page 147] implies that there exist $m_1,\ldots,m_r$ such that $o(g)=\mathrm{lcm}\{m_1,\ldots,m_r\}$, Φ_{m_i} divides $\mu_{\mathbb{Q},g}$, and $6=\sum_{i=1}^r d_i\phi(m_i)$, for some $d_i \geq 1$. In particular each $\phi(m_i) \leq 6$. Since $\pi(G) \subseteq \{2,3\}$, from a case-by-case analysis it follows that $m_i \in \{1,2,3,4,6,9,18\}$. In particular, $\exp(G) \mid 36$.

Suppose that G contains an element g of order 9. Then since $F(\zeta_9) \cong \frac{F[X]}{(\mu_{F,g})}$, and $\deg(\mu_{F,g})=2$,

$$|F(\zeta_9):\mathbb{Q}|=|F(\zeta_9):F||F:\mathbb{Q}|=6,$$

and $|\mathbb{Q}(\zeta_9):\mathbb{Q}|=6$, it follows that $F(\zeta_9)=\mathbb{Q}(\zeta_9)$, and in particular $F \subseteq \mathbb{Q}(\zeta_9)$. The only subfields contained in $\mathbb{Q}(\zeta_9)$ are $\mathbb{Q}$, $\mathbb{Q}(\zeta_3)$ and $\mathbb{Q}(\zeta_9+\zeta_9^{-1})$, a contradiction, and of these only $\mathbb{Q}(\zeta_9+\zeta_9^{-1})$ is of degree 3 over $\mathbb{Q}$. In particular, we conclude that if $\pi(G) \subseteq \{2,3\}$ and elements of order 3^n necessarily have order 3, then $\exp(G) \mid 12$. Suppose we are in this case, and let $e \in \mathrm{PCI}(G)$ such that $\mathbb{Q}Ge \cong \mathrm{M}_2(F)$. Then by Brauer's splitting field theorem[12], $\mathbb{Q}(\zeta_{12})$ is a splitting field for G, since G has exponent a divisor of 12. In particular, $F \subseteq \mathbb{Q}(\zeta_{12})$. However, $\mathrm{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$, which has order coprime to 3, which together with the fundamental theorem of Galois theory implies a contradiction. ☐

Now we are able to prove Theorem 3.7.

Proof of Theorem 3.7. First suppose that G has (M_{exc}), then by Theorem 2.1 the only division algebra components of $\mathbb{Q}G$ are of the form **XX**. Hence inspecting Proposition 3.5 and Remark 3.6 we see that all components indeed have the stated virtual cohomological dimension.

Conversely, if $\mathrm{vcd}(\mathrm{SL}_1(\mathbb{Q}Ge)) \in \{0,1,2,4\}$ for all $e \in \mathrm{PCI}(\mathbb{Q}G)$ then, by the results referred to above, the only simple algebras not allowed by the property (M_{exc}) are those of the form $\mathrm{M}_2(F)$, with F a cubic number field with one real embedding and ne pair of complex embeddings. However by Lemma 3.8 this can't be the simple component of $\mathbb{Q}G$. ☐

3.3. **Higher Kleinian groups: discrete subgroups of** $\mathrm{SL}_4(\mathbb{C})$**.** Another interesting property is a kind of higher Kleinian property:

Definition 3.10. A group Γ is said to *have property* Di_n if it is a discrete subgroup of $\mathrm{SL}_n(\mathbb{C})$, but not of $\mathrm{SL}_{n-1}(\mathbb{C})$.

We will be interested in the case that Γ has Di_n with n a divisor of 4. An alternative way to look at this is via the 5-dimensional hyperbolic space, as one has the following isomorphism:

$$\mathrm{Iso}^+(\mathbb{H}_5) \cong \mathrm{PGL}_2(\left(\frac{-1,-1}{\mathbb{Q}}\right))$$

In particular a group Γ acts discontinuously **correct terminology?** on[13] $\mathbb{H}_5$ if and only if Γ has Di_4 .

The finite dimensional simple algebras A such that $\mathrm{SL}_1(A)$ is Kleinian, i.e. is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ were classified in [15, Proposition 3.2].

[12]referentie?

[13]Hereby one assumes that the action on $\mathbb{H}_5$ doesn't come from the inbedding of an action on $\mathbb{H}_4$.

Proposition 3.11. *Let A be a finite dimensional simple F-algebra with F a number field and $\mathcal{O}$ an order in A. Then $\mathrm{SL}_1(A)$ has property Di_4 if and only if A has one of the following forms:*

(1) $\mathrm{M}_2(\left(\frac{-a,-b}{\mathbb{Q}}\right))$ *with* $a, b \in \mathbb{N}_0$,
(2) $\mathrm{M}_4(\mathbb{Q})$,
(3) $\left(\frac{-a,-b}{\mathbb{Q}(\sqrt{-d})}\right)$ *with* $a, b \in \mathbb{N}_0$ *and* $d \in \mathbb{N}_{>1}$ *square-free,*
(4) a division algebra of degree 4 which is non-ramified at at most one infinite place.

Proof. Dit ging via een karakterisatie als discreet inbedden in product plaatsen en dan strong approximation. Vervolgens ook nog aantonen dimensie deler van 16 of zo. ☐

Remark 3.12. The proof of Proposition 3.11 also yields that $\mathrm{SL}_1(\mathcal{O})$ has property Di_3 if and only if A is isomorphic to $\mathrm{M}_3(\mathbb{Q}), \mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ or a division algebra of degree 3 which is non-ramified at at most one infinite place.

Concerning Di_2, [15, Proposition 3.2] says that $\mathrm{SL}_1(\mathcal{O})$ has Di_2 if and only if $\mathrm{vcd}(A) \leq 2$ or A is quaternion division algebra which is ramified at all its infinite places and having exactly one pair of complex embeddings.

Explain, with Free-by-free in mind it is natural to put the condition that there is no exceptional type I

Theorem 3.13. *Let G be a finite group satisfying* $(\mathrm{M}_{\mathrm{exc}})$ *and having no exceptional division algebra components. Then G has Di_n for $n \mid 4$. The converse also holds if G has a fitting subgroup of index* 2.

Proof. Next consider property Di_n with n a divisor of 4. Combining Remark 3.12, Proposition 3.11 and Theorem 2.1 we see that $(\mathrm{M}_{\mathrm{exc}})$ implies the desired property. **To complete the converse once the statement is chosen** ☐

Remark 3.14. The condition that there is no exceptional division algebra components is an important one. Indeed, **The groups $C_3 \rtimes C_{2^n}$ and $C_m \times Q_8$ have** $(\mathrm{M}_{\mathrm{exc}})$ **but not** Di_n **with $n \mid 4$ due to bad division components!!!** In particular we see that in Free-by-free, i.e. when all components are even nicer, then also the condition is required. Consider the group XX. This has Di_n for $n \mid 4$, but no $(\mathrm{M}_{\mathrm{exc}})$. For that group however the fitting subgroup has index 4.

3.4. **Further questions.** blabla

4. The good property and amalgamated products of exceptional components

4.1. **Amalgams of Higher Bianchi groups. In this section we prove, using our previous works, that the 2×2 exceptional components are virtually of the desired amalgam type. We zullen hier sowieso nog verdere studie van de amalgams van onze werk in literatuur moeten doen.**

Proposition 4.1. *Let $\mathcal{O}$ be an order in D such that $\mathcal{U}(\mathcal{O})$ is finite. Then $E_2(\mathcal{O})$ has virtually such an amalgam.*

4.2. **The good property. The aim of this section will be to give several equivalent geometric group theoretical properties for** $(\mathrm{M}_{\mathrm{exc}})$

For this section we fix for each $e \in \mathrm{PCI}(G)$ a maximal order $\mathrm{M}_{n_e}(\mathcal{O}_e)$ in $\mathbb{Q}Ge$. The arguments will however be independent of this choice. Further denote by $\widehat{\Gamma}$ the profinite completion of a group Γ. Recall that Γ is called *good* if the map

$$H^j(\widehat{\Gamma}, M) \to H^j(\Gamma, M),$$

induced by the inclusion of Γ in its completion, is an isomorphism for any j and finite Γ-module M. By [5] the property to be good is one of commensurability classes and is closed under direct product. In particular $\mathrm{SL}_1(\mathbb{Z}G)$ is good exactly when $\mathrm{SL}_n(\mathcal{O}_e)$ is good for all

$e \in \mathrm{PCI}(G)$. **Still to clarify role of the center!!** Combining results in the literature one obtains:

Lemma 4.2. *Let G be a finite group with* $(\mathrm{M}_{\mathrm{exc}})$. *Then* $\mathrm{SL}_1(\mathbb{Z}Ge)$ *satisfies the good property for all* $e \in \mathrm{PCI}(\mathbb{Q}G)$.

Proof. In ?? it was proven that $\mathrm{SL}_n(\mathcal{O})$ is not good if it enjoys the subgroup congruence property. In particular if $n \geq 3$ or $n = 2$ and $\mathcal{U}(\mathcal{O})$ is infinite, then it is not good, e.g. see ??.

Put Zaleskii stuff together for the matrix components. But for quaternion algebras really need to think!! □

Using this we know can solve the virtual structure problem for $\mathcal{U}(\mathbb{Z}G)$ and the good property.

Theorem 4.3. *Let G be a finite group. Then $\mathcal{U}(\mathbb{Z}G)$ is good if and only if een van de karakterisaties van de vorige secties.*

5. The blockwise Zassenhaus property

An interesting corollary of **??** is that the finite groups such that $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_\infty$ satisfy a kind of component-wise third Zassenhaus conjecture.**to update to now notations**

Corollary 5.1. *Let G be a finite group such that $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_\infty$. If H is a finite subgroup of $V(\mathbb{Z}G)$, then for all $e \in PCI(\mathbb{Q}G)$ we have that He is conjugated, inside $\mathcal{U}(\mathbb{Z}Ge)$, to a finite subgroup of Ge.*

Recall that $\mathcal{U}(\mathbb{Z}G) = \pm 1.V(\mathbb{Z}G)$ where $V(\mathbb{Z}G)$ are the units of augmentation one and that $\mathcal{U}(\mathbb{Z}G))$ satisfies the third Zassenhaus conjecture if any $H \leq V(\mathbb{Z}G)$ is conjugated inside $\mathbb{Q}G$ to a subgroup of G. Using the recent survey [**?**] one can check that the 12 families of groups satisfying virtually-$\mathcal{G}_\infty$ (see [18, Theorem 1]) are not all among a known case of the third Zassenhaus conjecture. In case of the Zassenhaus conjectures the conjugation is expected to be in $\mathcal{U}(\mathbb{Q}G)$, hence it is remarkable that for this class of groups one can perform the conjugation inside an order of $\mathcal{U}(\mathbb{Q}Ge)$.

The proof of Corollary 5.1 will in fact be a corollary of Lemma 5.2 and the investigation of independant interest of the "Strong Zassenhaus property", which we introduce in Section 5.1, for exceptional components.

5.1. Zassenhaus property for semisimple algebras. write my notes here

To proof Corollary 5.1 we first record the following lemma which is of independent interest. **Make following more general !!**

Lemma 5.2. *Let G be a finite group and $H \leq V(\mathbb{Z}G)$ a finite subgroup. Then $|He| \mid |Ge|$ for every primitive central idempotent e of $\mathbb{Q}G$.***also prove exponente and set of order elements**

Proof. Fix a primitive central idempotent e of $\mathbb{Q}G$ and consider the associated epimorphism $\varphi : G \to Ge$. We $\mathbb{Z}$-linearly extend the latter to the ring epimorphism $\Phi : \mathbb{Z}[G] \to \mathbb{Z}[Ge]$. Denote $N := \ker(\varphi) = \{g \in G \mid ge = e\}$. Note that $\ker(\Phi) = \omega(G, N)$, the relative augmentation ideal. Also, by definition of the map, $\Phi(V(\mathbb{Z}[G])) \subseteq V(\mathbb{Z}[Ge])$. Hence $\Phi(H)$ is a finite subgroup of $V(\mathbb{Z}[Ge])$ hence $|\Phi(H)| \mid |Ge|$ (e.g. see [**?**, Corollary 2.7]).

It remains to prove that $|He| \mid |\Phi(H)|$. For this define the ring epimorphism $\pi : \mathbb{Z}[G] \to \mathbb{Z}[G]e : x \mapsto xe$. Since $\pi(n-1) = ne - e = 0$ for all $n \in N$, one has that $\omega(G, N) \subseteq \ker(\pi)$. Therefore we have a unique morphism $\sigma : \mathbb{Z}[Ge] \to \mathbb{Z}[G]e$ such that $\pi = \sigma\,\Phi$. In particular $He = \pi(H) = \sigma(\Phi(H))$ is an epimorphic image of $\Phi(H)$ and hence $|He| \mid |\Phi(H)|$, as needed. □

5.2. **Strong Zassenhaus for quaternion algebras and fields.**

Proposition 5.3. *Let G be a finite group and $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that $\mathbb{Q}Ge$ is some quaternion algebra or a field. Then for any $H \leq \mathrm{V}(\mathbb{Z}G)$ the groups He and Ge are conjugated in $\mathbb{Q}Ge$.*

Proof. **TO COMPLETE to all quaternions** Suppose $\mathbb{Q}G_e$ is isomorphic to a field F. The unit group of its unique maximal order (i.e. its rings of integers) is a finitely generated abelian group. Thus He and Ge are subgroups of the torsion group which is cyclic. Hence the dividing orders yield that $He \leq Ge$, as desired. Suppose $\mathbb{Q}Ge \cong \left(\frac{-a,-b}{\mathbb{Q}}\right)$, then by [22, Theorem 11.5.14] $\mathcal{U}(\mathbb{Z}Ge)$ is cyclic except if $(a,b) = (-1,-1)$ or $(a,b) = (-1,-3)$ and $\mathbb{Z}Ge$ is the Lipschitz order, the Hurwitz order or the maximal order[14] of $\left(\frac{-1,-3}{\mathbb{Q}}\right)$. Recall that the last two cases are the unique maximal order, thus we already have that $He \leq Ge$ if $\mathbb{Z}Ge$ is not the Lipschitz order in $\left(\frac{-1,-1}{\mathbb{Q}}\right)$. In the remaining case $Ge \cong Q_8$ and He is some subgroup of the unit group of the Hurwitz quaternions (i.e. a subgroup of $\mathrm{SL}(2,3) \cong Q_8 \rtimes C_3$). Since $|He| \mid |Ge|$ we see that in fact $He \leq Ge$, finishing all possible cases. ☐

5.3. **The exceptional** $\mathrm{GL}_2(\mathcal{O})$ **case.** Do here like we did for $\mathrm{GL}_2(\mathbb{Z})$ in our previous paper.

Theorem 5.4. *Let G be a finite group and $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that $\mathbb{Q}Ge \cong \mathrm{M}_2(D)$ with*[15] *$D \in \{\mathbb{Q}(\sqrt{-d}), \left(\frac{-a,-b}{\mathbb{Q}}\right) \mid a,b,d \in \mathbb{N}\}$. Then for any $H \leq \mathrm{V}(\mathbb{Z}G)$ the groups He and Ge are conjugated in $\mathbb{Q}Ge$.*

plan: component per component werken via expliciete amalgam van 'grote' finite index deelgroepen. Probleem is dat niet steeds de hele component zo'n decompositie heeft, maar mischien $\mathbb{Z}Ge$ desondanks toch steeds in eentje?

Proof. If it is $\mathrm{M}_2(\mathbb{Q})$, then $\mathcal{U}(\mathbb{Z}Ge) = \mathrm{GL}_2(\mathbb{Z}) \cong D_8 *_{C_2 \times C_2} D_{12}$. Therefore there exists some $\alpha_e \in \mathcal{U}(\mathbb{Z}Ge)$ such that $\alpha_e^{-1} Ge \alpha_e$ is a subgroup of D_8 or D_{12}. Since the $\mathbb{Q}$-span of Ge is $\mathrm{M}_2(\mathbb{Q})$ we get that $\alpha_e^{-1} Ge \alpha_e = D_6, D_{12}$ or D_8. In particular $|Ge|$ determines uniquely its isomorphism type. The same holds for He if He is non-abelian. Therefore, as $|He| \mid |Ge|$ by Lemma 5.2, we have the desired statement in that case. Suppose now that $|He| = 4$. If $He \cong C_4$, then He is up to conjugation a subgroup of D_8 and due to the dividing of the orders, again $He \leq Ge$ after conjugation. If it is an elementary abelian 2-group, then it is uniquely defined in both D_{12} and D_8. As this subgroup is amalgamated we are done. ☐

Proof of Corollary 5.1. By Lemma 5.2 $|He| \mid |Ge|$ for every primitive central idempotent e of $\mathbb{Q}G$. By Theorem 2.1 the simple algebra $\mathbb{Q}Ge$ is either a field, a specific type of quaternion algebra or some exceptional simple algebras. The strong Zassenhaus property of the former is proven in ?? and for the latter in Theorem 5.4. **TO UPDATE**. ☐

5.4. **On the difference between block-wise ISO and ISO.** We develop an obstruction theory describing how to glue properties ZC-properties of components to one for the semisimple algebras. Hopefully we show that this obstruction vanishes if $\mathbb{Q}G$ embeds in its 1×1 and exceptional 2×2-components. Who know start some non-abelian cohomology theory that measures the difference (in the philosophy of Kimmerle-Roggenkamp)...

References

[1] S. A. Amitsur. Finite subgroups of division rings. *Trans. Amer. Math. Soc.*, 80:361–386, 1955. 5

[2] S. A. Amitsur. Groups with representations of bounded degree. II. *Illinois J. Math.*, 5:198–205, 1961. 13

[3] Andreas Bächle, Geoffrey Janssens, Eric Jespers, Ann Kiefer, and Doryan Temmerman. Abelianization and fixed point properties of units in integral group rings. *Math. Nachr.*, 296(1):8–56, 2023. 5, 9, 21

[4] Florian Eisele, Ann Kiefer, and Inneke Van Gelder. Describing units of integral group rings up to commensurability. *J. Pure Appl. Algebra*, 219(7):2901–2916, 2015. 5

[14] A short concrete summary of the above orders can also be found before and after Theorem 3.14. of [3].

[15] In other words $\mathbb{Q}Ge$ is an exceptional component of $\mathbb{Q}G$.

[5] F. Grunewald, A. Jaikin-Zapirain, and P. A. Zalesskii. Cohomological goodness and the profinite completion of Bianchi groups. *Duke Math. J.*, 144(1):53–72, 2008. 19
[6] B. Huppert. *Endliche Gruppen. I*, volume 134 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1967. 12
[7] Bertram Huppert and Olaf Manz. Degree-problems. I. Squarefree character degrees. *Arch. Math. (Basel)*, 45(2):125–132, 1985. 11
[8] I. M. Isaacs and D. S. Passman. A characterization of groups in terms of the degrees of their characters. II. *Pacific J. Math.*, 24:467–510, 1968. 10, 11, 12, 13
[9] I. Martin Isaacs. *Character theory of finite groups*. Dover Publications, Inc., New York, 1994. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)]. 10, 11
[10] Geoffrey Janssens and Leo Margolis. On integral decomposition of unipotent elements in integral group rings, 2023. 7, 13
[11] Geoffrey Janssens, Doryan Temmerman, and François Thilmany. Simultaneous ping-pong for finite subgroups of reductive groups, 2024. 14
[12] E. Jespers and W. Sun. Nilpotent decomposition in integral group rings. *J. Algebra*, 575:127–158, 2021. 6, 8
[13] Eric Jespers and Ángel del Río. *Group ring groups. Vol. 1. Orders and generic constructions of units*. De Gruyter Graduate. De Gruyter, Berlin, 2016. 6, 7, 8
[14] Eric Jespers and Ángel del Río. *Group ring groups. Vol. 1. Orders and generic constructions of units*. De Gruyter Graduate. De Gruyter, Berlin, 2016. 13
[15] Eric Jespers, Antonio Pita, Ángel del Río, Manuel Ruiz, and Pavel Zalesskii. Groups of units of integral group rings commensurable with direct products of free-by-free groups. *Adv. Math.*, 212(2):692–722, 2007. 4, 9, 12, 14, 15, 17, 18, 19
[16] E. Kleinert and Á. del Río. On the indecomposability of unit groups. *Abh. Math. Sem. Univ. Hamburg*, 71:291–295, 2001. 14
[17] Reginald Koo. A Classification of Matrices of Finite Order over $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$. *Math. Mag.*, 76(2):143–148, 2003. 17, 18
[18] Guilherme Leal and Angel del Río. Products of free groups in the unit group of integral group rings. II. *J. Algebra*, 191(1):240–251, 1997. 20
[19] Thomas Noritzsch. Groups having three complex irreducible character degrees. *J. Algebra*, 175(3):767–798, 1995. 11
[20] Sudarshan K. Sehgal. Nilpotent elements in group rings. *Manuscripta Math.*, 15:65–80, 1975. 7
[21] M. Shirvani and B. A. F. Wehrfritz. *Skew linear groups*, volume 118 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1986. 5, 6
[22] John Voight. *Quaternion algebras, v.0.9.13*. June 10, 2018. https://math.dartmouth.edu/~jvoight/quat/quat-book-v0.9.13.pdf. 21

(Robynn Corveleyn)
Institut de Recherche en Mathématique et Physique, UCLouvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium
Email address: robynn.corveleyn@uclouvain.be

(Geoffrey Janssens)
Institut de Recherche en Mathématique et Physique, UCLouvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium and
Department of Mathematics and Data Science, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Elsene, Belgium
Email address: geoffrey.janssens@uclouvain.be

(Doryan Temmerman)
Department of Mathematics and Statistics, Universiteit Hasselt, Martelarenlaan 42, 3500 Hasselt, Belgium
Email address: doryan.temmerman@uhasselt.be