SIMULTANEOUS PING-PONG FOR FINITE SUBGROUPS OF REDUCTIVE GROUPS

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Abstract. Let Γ be a Zariski-dense subgroup of a reductive group $\mathbf{G}$ defined over a field F. Given a finite collection of finite subgroups H_i $(i \in I)$ of $\mathbf{G}(F)$ avoiding the center, we establish a criterion to ensure that the set of elements of Γ that form a free product with every H_i (so-called ping-pong partners for H_i) is both Zariski- and profinitely dense in Γ. This criterion applies to direct products of inner $\mathbb{R}$-forms of GL_n with $n \geq 2$, and implies a particular case (the case of torsion elements in such products) of a conjecture of de la Harpe. Subsequently, we give constructive methods to obtain such ping-pong partners, again when $\mathbf{G}$ is a direct product of inner forms of GL_n for $n \geq 2$.

Next, we investigate the case where $\mathbf{G} = \mathcal{U}(FG)$ for G a finite group and $\Gamma = \mathcal{U}(RG)$ for R an order in F. Hereby we prove that the set of bicylic unit ping-pong partners of a given shifted bicyclic unit is profinitely dense in $\mathcal{U}(RG)$, answering a long standing common belief in the field. Finally, we answer the Virtual Structure Problem for the property to have an amalgam or HNN spliting over a finite group.

Contents

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1. Introduction

The construction and study of free products in linear groups is a classical topic going back to the early days of group theory. A groundbreaking progress was Tits' celebrated alternative [74] establishing existence of free subgroups in linear groups which are not virtually solvable. In fact he proved the stronger statement that if Γ is a finitely generated subgroup of $\mathrm{GL}_n(F)$ with F some field such that its Zariski closure is a Zariski connected semisimple algebraic group, then either Γ contains a Zariski open solvable subgroup or it contains a Zariski dense free subgroup of finite rank. The connected assumption on $\overline{\Gamma}$ was removed by Breuillard-Gelander [9] where the authors also obtain the same, but much stronger, statement for the topology coming from F in case it is a local field. The speed at which a given finite set Σ, such that $\langle \Sigma \rangle$ is not virtually solvable, produces a free subgroup was also clarified in a remarkably strong sense over the years, e.g. see [1, 10, 12, 11, 13, 2] for some recent results.

In this article we are interested in the problem of constructing free groups with a given fixed generator. More generally, given a finite subset F of a linear group Γ, the question of interest is: does there exists an element $y \in \Gamma$ such that $\langle x, y \rangle \cong \langle x \rangle * \langle y \rangle$? If yes, how large (in a topological sense) is the set of such elements y? These elements y are called *simultaneous ping-pong partners* of the set F. In 2007 de la Harpe expressed the following conjecture:

Conjecture 1.1 (de la Harpe, [16, Question 16]). *Let $\mathcal{G}$ be a connected semisimple real Lie group without compact factors, and let Γ be a Zariski-dense subgroup of the adjoint group $\mathrm{Ad}(\mathcal{G})$. Let F be a finite set of non-trivial elements of Γ. Does there exists a $\gamma \in \Gamma$ of infinite order such that $\langle h, \gamma \rangle \cong \langle h \rangle * \langle \gamma \rangle$ for every $h \in F$?*

In [69, Theorem 1.3] Soifer & Vishkautsan have given a positive answer in the case of $\mathrm{PSL}_n(\mathbb{Z})$ and that F only contains elements whose semisimple part is either hyperbolic or torsion. Furthermore they mention that the methods also work for a lattice Γ in $\mathrm{PSL}_n(k)$ with k a local field. In case that $\mathcal{G}$ does not contain simple factors of type A_n, D_{2n+1} or E_6, then a positive answer was claimed for any F in [60, Theorem 6.5]. Unfortunately this important (unpublished) preprint contains non-correctable errors, see Remark 3.15 for more details.

1. Criterion simultaneous ping-pong with finite subgroups and type A case. In the first half of the article we consider a slightly more general version of Conjecture 1.1 as we will allow $\mathcal{G}$ to be reductive and F to contain subgroups (and not only elements). More precisely we suppose that F is a finite set of finite subgroups of Γ. In that generality we obtain following sufficient conditions.

Theorem 3.2. *Let $\mathbf{G}$ be a connected algebraic F-group with center $\mathbf{Z}$. Let Γ be a Zariski-connected subgroup of $\mathbf{G}(F)$. Let $(H_i)_{i \in I}$ be a finite collection of finite subgroups of $\mathbf{G}(F)$, and set $C_i = H_i \cap \mathbf{Z}(F)$. Assume that for each $i \in I$ there exists a local field K_i containing F and a projective K_i-representation $\rho_i : \mathbf{G} \to \mathrm{PGL}_{V_i}$, where V_i is a finite-dimensional module over a finite division K_i-algebra D_i, with the following properties:*

(Proximality) *$\rho_i(\Gamma)$ contains a proximal element;*

(Transversality) *For every $h \in H_i \backslash C_i$ and every $p \in \mathbb{P}(V_i)$, the span of the set $\{\rho_i(xhx^{-1})p \mid x \in \Gamma\}$ is the whole of $\mathbb{P}(V_i)$.*

Let S be the collection of regular semisimple elements $\gamma \in \Gamma$ of infinite order, such that for all $i \in I$, the canonical map

$$(\langle \gamma \rangle \times C_i) *_{C_i} H_i \to \langle \gamma, H_i \rangle \leq \mathbf{G}(F)$$

is an isomorphism. Then S is dense in Γ for the join of the profinite topology and the Zariski topology.

See Section 3.2 for background on some classical projective dynamics (as definition of proximal elements), which is worked out over division algebras, and Section 2 for background on amalgamated products.

Next, in Section 3.4 we verify the proximality and transversality condition for finite subgroups in products of inner forms of SL_n and GL_n. As a consequence, by the above theorem, we establish the abundance of simultaneous ping-pong partners

Theorem 3.23. *Let $\mathbf{G}$ be a reductive $\mathbb{R}$-group whose simple quotients are each isogenic to PGL_{D^n} for D some finite division $\mathbb{R}$-algebra and $n \geq 2$, and let $\mathbf{Z}$ denote its center. Let Γ be a subgroup of $\mathbf{G}(\mathbb{R})$ whose image in $\mathrm{Ad}\,\mathbf{G}$ is Zariski-dense. Let $(H_i)_{i\in I}$ be a finite collection of finite subgroups of $\mathbf{G}(\mathbb{R})$, and set $C_i = H_i \cap \mathbf{Z}(\mathbb{R})$.*

Suppose that for each $i \in I$, there exists a simple quotient $\mathbf{Q}_i$ of $\mathbf{G}$ for which the kernel of the projection $H_i \to \mathbf{Q}_i(\mathbb{R})$ is contained in C_i. Then the collection of regular semisimple elements $\gamma \in \Gamma$ of infinite order such that for all $i \in I$, the canonical map

$$(\langle\gamma\rangle \times C_i) *_{C_i} H_i \to \langle\gamma, H_i\rangle \leq \mathbf{G}(\mathbb{R})$$

is an isomorphism, is dense in Γ for the join of the profinite topology and the Zariski topology.

Given a reductive F-group $\mathbf{G}$ with center $\mathbf{Z}$ and a subgroup $H \leq \mathbf{G}(F)$ we say that H *almost embeds in a (simple) quotient $\mathbf{Q}$ of $\mathbf{G}$* if there exists a (simple) quotient $\mathbf{Q}$ of $\mathbf{G}$ for which the kernel of the restriction $H \to \mathbf{Q}(F)$ is contained in $\mathbf{Z}(F)$. In Proposition 2.7 it is proven that this is a necessary condition for a finite subgroup to be part of a free product.

Despite that there is profinitely many simultaneous ping-pong partners, the above result, and its proof, does not answer how to construct such partners. For an algebraic group $\mathbf{G}$ which is a direct product of inner forms of GL_n for $n \geq 2$, we consider this problem in Section 4. More precisely, we tackle question when two given finite subgroups H and K can serve as parts of a direct product $H * K$. To do so we introduce in Section 4.1 the concept of a basic nilpotent transformation, which intuitively is a linear deformation of K to reach the necessary ping-pong dynamics. The main result is Theorem 4.12.

2. The case of semisimple algebras and the unit group of a group ring Let A be a finite dimensional semisimple algebra over F. By the theorem of Wedderburn-Artin

$$A \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_1}(D_1).$$

as ring where D_i is a finite dimensional division algebra over F. In particular, $\mathbf{G} = \mathcal{U}(A)$ is a reductive group. Furthermore if $\mathcal{O}$ is an order in A, then by classical results of Borel and Harish-Chandra $\Gamma = \mathcal{U}(\mathcal{O})$ is an arithmetic subgroup of $\mathcal{U}(A)$ and in particular one is in the setting Theorem 3.23. A non-trivial consequence of the latter is following necessary and sufficient conditions for amalgamted producrts in $\mathcal{U}(\mathcal{O})$.

Corollary 4.1. *Let F be a number field, A be a finite semisimple F-algebra, and $\mathcal{O}$ be an order in A. Let Γ be a Zariski-dense subgroup of $\mathcal{U}(\mathcal{O})$. Let H be a finite subgroup of $\mathcal{U}(A)$, and C its intersection with the center of A.*

There exists $\gamma \in \Gamma$ of infinite order with the property that the canonical map

$$(\langle\gamma\rangle \times C) *_C H \to \langle\gamma, H\rangle$$

is an isomorphism, if and only if H almost embeds in Ae for some $e \in \mathrm{PCI}(A)$ for which Ae is neither a field nor a totally definite quaternion algebra.

Moreover, in the affirmative, the set of such elements γ is dense in the join of the Zariski and the profinite topology.

If $A = FG$ is a group algebra, then the tuple $(\mathrm{M}_{n_1}(D_1), \cdots, \mathrm{M}_{n_1}(D_1))$ of simple components of A is not at all arbitrary. The reason for this is that $\mathrm{span}_F(Ge_i) = \mathrm{M}_{n_i}(D_i)$ for all i, where e_i is the primitive central idempotent of FG assocaited to $\mathrm{M}_{n_i}(D_i)$. Thanks to this, one can try to use finite group (representation) theory to determine when a finite subgroup $H \leq \mathcal{U}(\mathcal{O})$ enjoys the almost-embedding condition. This problem is the content of Section 5.3.1 and the main result is the following.

Theorem 5.16 & Corollary 5.20. *Let F be a number field and R its rings of intgers. Further let G be a finite group and $h \in \mathcal{U}(RG)$ torsion. Suppose that one of the following cases hold:*

(I) $h^{\alpha} \in \mathcal{U}(R) \,.\, G$ for some $\alpha \in FG$.

(II) $o(h)$ is a prime power.

If $\langle h \rangle \cap \mathcal{Z}(G) = 1$, then there exists some $e \in PCI(FG)$ such that $\langle h \rangle \cap \ker(\pi_e) = 1$ and FGe is neither a field nor a totally definite quaternion algebra. Consequently, there exists some $t \in \mathcal{U}(RG)$ such that

$$\langle h, t\rangle \cong \langle h \rangle *_C \langle t, C\rangle \cong C_{o(h)} *_C (\mathbb{Z} \times C),$$

where $C = \langle h \rangle \cap \mathcal{Z}(G)$.

The above results in particular also yields a new proof of the main existence result of $C_p * \mathbb{Z}$ in $\mathcal{U}(\mathbb{Z}G)$ by Goncalves-Passman [27]. Also see Remark 5.17 on when conditions (I) and (II) hold. In particular note that condition (I) is reminiscent of the first Zassenhaus conjecture.

Next, we consider the problem of forming a free product with a specific type of unit in $\mathcal{U}(RG)$. More precisely, consider in RG the elements of the form

$$b_{\tilde{h},x} = 1 + (1-h)x\widetilde{h} \text{ and } b_{x,\tilde{h}} = 1 + \widetilde{h}x(1-h)$$

with $x \in RG$ and $\widetilde{h} := \sum_{i=1}^{o(h)} h^i$. All these elements are unipotent units in $\mathcal{U}(RG)$. The elements in the group

$$\mathrm{Bic}(G) := \langle b_{\tilde{h},x}, b_{x,\tilde{h}} \mid x \in RG\rangle$$

are called *bicyclic units*. For more than 20 years it has been a conjecture in the field of group rings that two generically chosen bicyclic units generate a free group. In this claim the word 'generic' has however never been made precise. Our next mail result shows that this long standing problem is correct for the profinite topology (and hence in the Zariski topology), modulo the minor subtility that one needs to slightly deform the given bicyclic unit to $b_{\tilde{h},x}h = h + (1-h)x\widetilde{h}$. An element of the latter form is called *shifted bicyclic unit* in the literature.

Theorem 5.8. *Let $H \leq G$ be finite groups and $\alpha = 1 + (1-h)x\widetilde{H}$ a bicyclic unit for some $h \in H$ and $x \in RG$. Then*

$$\mathcal{P}(\alpha) := \{\beta \in \mathrm{Bic}(G) \mid \langle \alpha h, \beta\rangle \cong \langle \alpha h\rangle * \langle \beta \rangle\}$$

is a profinite dense subset in $\mathrm{Bic}(G)$.

3. The virtual structure problem for a product of amalgam or HNN over finite groups Finally, we consider the the Virtual Structure Problem, which askes for a unit theorem. A very concrete idea of a unit theorem was given by Kleinert [49] in the context of orders:

> *A unit theorem for a finite dimensional semisimple rational algebra A consists of the definition, in purely group theoretical terms, of a class of groups $C(A)$ such that almost all generic unit groups of A are members of $C(A)$.*

Recall that a generic unit group of A is a subgroup of finite index in the group of reduced norm 1 elements of an order in A. Till recently, the finite groups G for which a unit theorem, in the sense of Kleinert, was known for $\mathcal{U}(\mathbb{Z}G)$ are those for which the class of groups considered are either finite groups (Higman), abelian groups (Higman), or direct products of free-by-free groups [42, 39, 58, 44]. Remarkably, the latter class can also be described in terms of the rational group algebra: every simple quotient of $\mathbb{Q}G$ is either a field, a totally definite quaternion algebra or $\mathrm{M}_2(K)$, where K is either $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3})$. Such type of unit theorem was also obtained recently in [4, 3] for several geometric properties such as property (T) and (HFA). To our knowledge this result covers all the known unit theorems on $\mathcal{U}(\mathbb{Z}G)$.

In this article we answer the above problem in the case of having infinitely many ends, i.e. consider the following class of groups:

$$\mathcal{G}_\infty := \{\prod_i \Gamma_i \mid \Gamma_i \text{ has infinitely many ends } \}.$$

By Stallings theorem [71, 70] a group has infinitely many ends if and only if it can be decomposed as an amalgamated product or HNN extension over a finite group. In fact we will mainly work with this characterisation.

Theorem 6.2. *Let G be a finite group. The following are equivalent:*

(1) $\mathcal{U}(\mathbb{Z}G)$ is virtually in $\mathcal{G}_\infty$,

(2) all the simple components of $\mathbb{Q}G$ are of the form $\mathbb{Q}(\sqrt{-d})$, with $d \in \mathbb{N}$, $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ with non-zero $a, b \in \mathbb{N}$ or $\mathrm{M}_2(\mathbb{Q})$ and the latter needs to occur.

(3) $\mathcal{U}(\mathbb{Z}G)$ is virtually a direct product of non-abelian free groups

Moreover, only the parameters $(-1,-1)$ and $(-1,-3)$ can occur for $(-a,-b)$. Also, $e(\mathcal{U}(\mathbb{Z}G)) = \infty$ if and only if it virtually free if and only if G is isomorphic to $D_6, D_8, Dic_3, C_4 \rtimes C_4$.

The finite groups satisfying (3) in Theorem 6.2 have been classified in [39] and hence the result indeed answers the Virtual Structure problem for the property $\mathcal{G}_\infty$.

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2. Amalgams in almost-direct products

In this section, we recall a variant for amalgamated products of the classical ping-pong lemma. Thereafter we exhibit a necessary condition for a subgroup of an almost-direct product to be an amalgamated product.

Given a subgroup C of a group G, we will denote by T_C^G a set of representatives of the left cosets of C in G, containing the identity element.

The ping-pong lemma for amalgams and its variant for HNN extensions can be found in [51, Propositions 12.4 & 12.5]. For the convenience of the reader, we provide a proof as it will be instrumental in the rest of this paper.

Lemma 2.1 (Ping-pong for amalgams)**.** *Let A, B be subgroups of a group G and suppose $C = A \cap B$ satisfies $|A : C| > 2$. Let G act on a set X. If $P_1, P_2 \subset X$ are two subsets with $P_1 \not\subset P_2$, such that for all elements $a \in T_C^A \setminus \{e\}$, $b \in T_C^B \setminus \{e\}$ and $c \in C$, we have*

$$aP_1 \subset P_2, \quad bP_2 \subset P_1, \quad cP_1 \subset P_1, \quad \text{and} \quad cP_2 \subset P_2,$$

then the canonical map $A *_C B \to \langle A, B\rangle$ is an isomorphism.

As in the case of free products, the proof of Lemma 2.1 is straightforward once one knows the normal form for elements in an amalgamated product. The normal form also allows us to unambiguously speak of *words starting with A* and *words starting with B*. In the next lemma, these are the elements for which $\dot{a}_1 \notin C$, resp. for which $\dot{a}_1 \in C$.

Lemma 2.2 (Normal form in amalgams)**.** *Let $A, B \leq G$ be groups and $C \leq A \cap B$. The following are equivalent.*

(i) *The canonical map $A *_C B \to \langle A, B\rangle$ is an isomorphism.*

(ii) *Every element in $\langle A, B\rangle$ has a unique decomposition of the form $\dot{a}_1 b_1 \cdots a_n \dot{b}_n c$, where $a_i \in T_C^A \setminus \{e\}$, $b_i \in T_C^B \setminus \{e\}$, $\dot{a}_1 \in T_C^A$, $\dot{b}_n \in T_C^B$, and $c \in C$.*

(iii) *Given $a_i \in A \setminus C$, $b_i \in B \setminus C$, $\dot{a}_1 \in A$, and $\dot{b}_n \in B$, the product $\dot{a}_1 b_1 \cdots a_n \dot{b}_n$ belongs to C only if $n = 1$ and $\dot{a}_1, \dot{b}_n \in C$.*

In consequence of the affirmative, $C = A \cap B$.

Sketch of proof. The implication (i) $\Longrightarrow$ (ii) is the existence and uniqueness of a normal form (see for instance [67, Theorem 1]), and its converse amounts to checking the injectivity of the canonical map, which follows from the uniqueness of the decomposition in $\langle A, B\rangle$.

After replacing $\dot{b}_n, a_n, \ldots, b_1, \dot{a}_1$ by the appropriate coset representatives, (ii) $\Longrightarrow$ (iii) becomes obvious. For the contrapositive of its converse, note that two different decompositions of an element in $\langle A, B\rangle$ result in a non-trivial expression of the form $\dot{a}_1 b_1 \cdots a_n \dot{b}_n$ in C. □

Proof of Lemma 2.1. Note that the assumptions imply that $aP_1 \subset P_2$ for all $a \in A \setminus C$, $bP_2 \subset P_1$ for all $b \in B \setminus C$, and $cP_1 = P_1$, $cP_2 = P_2$ for every $c \in C$.

Suppose that given $a_i \in A \setminus C$, $b_i \in B \setminus C$, $\dot{a}_1 \in A$ and $\dot{b}_n \in B$, the non-empty word $c = \dot{a}_1 b_1 \cdots a_n \dot{b}_n$ lies in C. The possible cases for $\dot{a}_1$ and $\dot{b}_n$ to belong to C are:

- $\dot{a}_1 \notin C$, $\dot{b}_n \in C$. We have $\dot{b}_n P_1 = P_1$, $a_n \dot{b}_n P_1 \subset P_2$, $b_{n-1} a_n \dot{b}_n P_1 \subset P_1$, etc., so that eventually, $cP_1 = \dot{a}_1 b_1 \cdots a_n P_1 \subset P_2$. Since $cP_1 = P_1$ and $P_1 \not\subset P_2$, this case cannot occur.
- $\dot{a}_1 \in C$, $\dot{b}_n \notin C$. Pick $a \in A \setminus C$, and let $a' \in A$ and $c' \in C$ be such that $a^{-1}ca = a'c'$. We have $aa' \notin C$, hence the word $c' = (aa')^{-1} b_1 \cdots a_n \dot{b}_n a$ starts and ends with an element of $A \setminus C$. This case thus reduces to the first one.
- $\dot{a}_1 \notin C$, $\dot{b}_n \notin C$. As $|A : C| > 2$, we may pick $a \in A \setminus (C \cup \dot{a}_1 C)$, so that $a^{-1}\dot{a}_1 P_1 \subset P_2$ hence $\dot{a}_1 P_1 \subset aP_2$. As in the first case, we have $cP_2 \subset \dot{a}_1 P_1$. Since $cP_2 = P_2$, this would imply $aP_1 \subset P_2 \subset aP_2$, hence this case does not occur either.
- $\dot{a}_1 \in C$, $\dot{b}_n \in C$. If $n > 1$, replacing c by c^{-1} reduces to the third case. The only remaining possibility is thus $n = 1$ and $\dot{a}_1, \dot{b}_n \in C$, as expected.

We conclude from Lemma 2.2 that the canonical map $A *_C B \to \langle A, B\rangle$ is an isomorphism. □

Lemma 2.3. *Let $A *_C B$ be a free amalgamated product. If f is a surjective morphism from a group Γ to $A *_C B$, then Γ is the free product with amalgamation $f^{-1}(A) *_{f^{-1}(C)} f^{-1}(B)$.*

If moreover Γ is generated by two subgroups Γ_1, Γ_2 with the properties $f(\Gamma_1) \subseteq A$, $f(\Gamma_2) \subseteq B$, the induced map $\Gamma_1 \to A/C$ is injective, and $\Gamma_1(\Gamma_2 \cap f^{-1}(C))$ is a subgroup, then $f^{-1}(B) = \Gamma_2$ and $\Gamma \cong (\Gamma_1 f^{-1}(C)) *_{f^{-1}(C)} \Gamma_2$.

Proof. The first part of the lemma is standard (see for instance [78, Lemma 3.2]). For the second part, let $g = b_0 a_1 b_1 \cdots a_n b_n$ with $a_i \in \Gamma_1 \setminus \{e\}$ and $b_i \in \Gamma_2$ be an element of $f^{-1}(B)$. Since $(\Gamma_2 \cap f^{-1}(C))\Gamma_1 = \Gamma_1(\Gamma_2 \cap f^{-1}(C))$, after perhaps reducing the expression for g, we may assume that $b_i \notin f^{-1}(C)$ for $0 < i < n$. Because $f(\Gamma) = A *_C B$ and $f(b_i) \in B \setminus C$, Lemma 2.2 implies that $n = 0$, hence $g = b_0 \in \Gamma_2$. Thus $f^{-1}(B) = \Gamma_2$, and in consequence $f^{-1}(C) \leq \Gamma_2$. On the other hand, if $g = b_0 a_1 b_1 \cdots a_n b_n \in f^{-1}(A)$, we may assume as before that $a_i \neq e$ for $1 \leq i \leq n$ and $b_i \notin f^{-1}(C)$ for $0 < i < n$. Applying f again then shows that $n \leq 1$ and $b_i \in f^{-1}(C)$ for $i \leq n$, so that $g \in \Gamma_1 f^{-1}(C)$. □

The following folkloric terminology is inspired by Lemma 2.1.

Definition 2.4. Let A and B be subgroups of a group G. We say that A is *a ping-pong partner for B*, or that *A and B play ping-pong*, if the subgroup $\langle A, B\rangle$ is freely generated by A and B, or in other words if the canonical map $A * B \to \langle A, B\rangle$ is an isomorphism. Similarly, we say that $a \in A$ is *a ping-pong partner for B in A*, or that *a and B play ping-pong*, if the subgroup $\langle a, B\rangle$ is freely generated by $\langle a\rangle$ and B. When B is generated by a single element b, we also say that *a is a ping-pong partner for b*.

Sets P_1 and P_2 to which one can apply Lemma 2.1 are sometimes called a *ping-pong table for A and B*.

In the subsequent sections, we will look to play ping-pong inside a group $G = \prod_{i=1}^n G_i$ which decomposes into a direct product of subgroups G_i. Using some simple facts about free (amalgamated) products, the next proposition will show that this requires an embedding of the ping-pong partners in one of the factors G_i.

Given subgroups $H_1, \ldots, H_n$ of a group G, let $[H_1, \ldots, H_n] = [H_1, [H_2, \ldots, H_n]]$ denote the *left-iterated (or right-normed) commutator subgroup* of the H_i.

Lemma 2.5. *Let N, N_1, ..., N_n be normal subgroups of $A *_C B$, where $|A : C| > 2$.*

(i) *Either $N \subset C$, or N contains a non-abelian free group.*

(ii) *If $[N_1, N_2] \subset C$, then either $N_1 \subset C$ or $N_2 \subset C$.*

In consequence, if $[N_1, \ldots, N_n]$ admits no non-abelian free subgroups, there exists $i \in \{1, \ldots, n\}$ for which $N_i \subset C$.

Proof. First, suppose that N is a normal subgroup of $A *_C B$ not contained in C. Pick $x \in N \setminus C$; by Lemma 2.2, we may assume after conjugation that x either belongs to $B \setminus C$, belongs to $A \setminus C$, or is cyclically reduced starting with $a_1 \in A \setminus C$.

- If $x \in B \setminus C$, pick $a, a' \in A \setminus C$ such that $a \notin a'C$. Using Lemma 2.2, one readily checks that the cyclically reduced words $w = [a, x]$ and $w' = [a', x]$ generate a free group, as every non-empty word in w and w' remains a non-empty word alternating in elements of $A \setminus C$ and $B \setminus C$. (Only simplifications of the form $[a, x][a', x]^{-1} = ax(a^{-1}a')x^{-1}a'^{-1}$ occur, and the condition on a and a' ensures no further cancellations arise.)
- If $x \in A \setminus C$, pick $b \in B \setminus C$ and $a, a' \in A \setminus C$ such that $a \notin a'C$, and consider $w = [x, bab^{-1}]$ and $w' = [x, ba'b^{-1}]$ instead.
- In the last case, write $x = a_1 b_1 \cdots, a_n b_n$ with $n \geq 1$ and $a_i \in A \setminus C$, $b_i \in B \setminus C$. Pick $b \in B \setminus C$ and $a \in A \setminus C$ such that $a \notin a_1 C$. Then the words $w = x$ and $w' = aba^{-1}xab^{-1}a^{-1}$ generate a free group.

This proves part (i).

Second, suppose that there exist elements $x \in N_1 \setminus C$ and $x' \in N_2 \setminus C$. By Lemma 2.2, we may assume after conjugation that x, x' either belong to $A \setminus C$, belongs to $B \setminus C$, or is cyclically reduced starting with A. We exhibit in each case a commutator in $[N_1, N_2] \setminus C$.

- If $x = a_1$ and $x' = b_1'$, then $[x, x'] \notin C$.
- If x is cyclically reduced starting with a_1 and $x' = a_1'$, then $[x, bx'b^{-1}] \notin C$ for any $b \in B \setminus (C \cup b_n^{-1}C)$.
- If x is cyclically reduced starting with a_1 and $x' = b_1'$, then $[a^{-1}xa, x'] \notin C$ for any $a \in A \setminus (C \cup a_1 C)$.

- If $x = a_1$ and $x' = a'_1$, then $[x, bx'b^{-1}] \notin C$ for any $b \in B \setminus C$.
- If $x = b_1$ and $x' = b'_1$, then $[axa^{-1}, x'] \notin C$ for any $a \in A \setminus C$.
- If x, x' are both cyclically reduced starting with a_1, and ending with $b'_{n'}$ respectively, then $[a^{-1}xa, b^{-1}x'b] \notin C$ for any $a \in A \setminus (C \cup a_1 C)$ and $b \in B \setminus (C \cup b'^{-1}_{n'} C)$.

This proves part (ii).

Lastly, if $[N_1, \ldots, N_n]$ admits no non-abelian free subgroups, we deduce from part (i) that $[N_1, \ldots, N_n] \subset C$. Part (ii) then implies that either $N_1 \subset C$, or $[N_2, \ldots, N_n] \subset C$, and recursively, that eventually $N_i \subset C$ for some $i \in \{1, \ldots, n\}$. □

Definition 2.6. Let $\mathcal{S}$ be a class of groups closed under taking subquotients and extensions. For the purposes of the following proposition, we will say that G is an *$\mathcal{S}$-almost direct product of* $G_1, \ldots, G_n$ if G has a normal subgroup $K \in \mathcal{S}$ such that G/K is the direct product $G_1 \times \cdots \times G_n$.

Equivalently, if there are normal subgroups $M_1, \ldots, M_n$ of G such that $\bigcap_{i=1}^n M_i \in \mathcal{S}$ and $M_i(M_{i+1} \cap \cdots \cap M_n) = G$ for $i = 1, \ldots, n-1$, then G is the $\mathcal{S}$-almost direct product of $G/M_1, \ldots, G/M_n$. Indeed, the second condition ensures that the canonical map $G/\bigcap_{i=1}^n M_i \to G/M_1 \times \cdots \times G/M_n$ is surjective; conversely, writing M_i for the kernel of $G \to G_i$, it is obvious that $K = \bigcap_{i=1}^n M_i$ and $M_j(\bigcap_{i \neq j} M_i) = G$.

Almost direct products with respect to the class containing only the trivial group are just direct products. In the literature, almost direct products appear most often for $\mathcal{S}$ the class of finite groups. Here are a few straightforward observations:

- Any group in $\mathcal{S}$ is an $\mathcal{S}$-almost empty direct product; so of course the notion is meaningful only for groups outside of $\mathcal{S}$.
- An $\mathcal{S}$-almost direct product of groups $G_1, \ldots, G_n$ themselves $\mathcal{S}$-almost direct products of respectively $H_{i1}, \ldots, H_{in_j}$ ($i = 1, \ldots, n$), is an $\mathcal{S}$-almost direct product of the H_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n_j$.
- Any quotient or extension of an $\mathcal{S}$-almost direct product by a group in $\mathcal{S}$ is again an $\mathcal{S}$-almost direct product.

Sometimes, almost direct products are defined by the following variant: G is the quotient of a direct product $G_1 \times \cdots \times G_n$ by a normal subgroup $H \in \mathcal{S}$. An almost direct product in this second sense is also an $\mathcal{S}$-almost direct product in the sense of Definition 2.6. Indeed, if $G = (G_1 \times \cdots \times G_n)/H$, denoting π_i the projection onto G_i and $K = \pi_1(H) \times \cdots \times \pi_n(H)$, we see that $G/(K/H) \cong (G_1 \times \cdots \times G_n)/K = G_1/\pi_1(H) \times \cdots \times G_n/\pi_n(H)$. The converse however does not always hold, as the images of the factors G_i in $(G_1 \times \cdots \times G_n)/H$ are commuting normal subgroups, and this may not happen in G even if G/K is a direct product.

Proposition 2.7 (Amalgams in almost direct products)**.** *Let $\mathcal{S}$ be the class of groups not containing a non-abelian free group. Let G be the $\mathcal{S}$-almost direct product of groups $G_1, \ldots, G_m$, and suppose that $G_{n+1}, \ldots, G_m$ belong to $\mathcal{S}$. If A and B are subgroups of G whose intersection C satisfies $|A : C| > 2$, and are such that the canonical map $A *_C B \to \langle A, B \rangle$ is an isomorphism, then there exists $i \in \{1, \ldots, n\}$ for which the kernel of the projection $\langle A, B \rangle \to G_i$ is contained in C.*

Proof. Since $G_{n+1}, \ldots, G_m$ belong to $\mathcal{S}$, it is clear that G is also the $\mathcal{S}$-almost direct product of $G_1, \ldots, G_n$. Let π_i denote the projection $G \to G_i$ and set $M_i = \ker \pi_i$. Identify $\langle A, B \rangle$ with $A *_C B$ and set $N_i = M_i \cap (A *_C B)$.

By assumption, $\bigcap_{i=1}^n M_i$ does not contain a non-abelian free group. The same then holds for $[N_1, \ldots, N_n] \subset [M_1, \ldots, M_n] \subset \bigcap_{i=1}^n M_i$, and Lemma 2.5 implies the existence of $i \in \{1, \ldots, n\}$ for which $N_i \subset C$. □

There are versions of Lemma 2.5 and Proposition 2.7 for HNN extensions. We leave their statement and proof to the reader.

3. Simultaneous ping-pong partners for finite subgroups of reductive groups

Let F be a field. Let $\mathbf{G}$ be a reductive[1] algebraic F-group, Γ a Zariski-dense subgroup of $\mathbf{G}(F)$, and H a finite subgroup of $\mathbf{G}(F)$. This section is concerned with finding elements γ of Γ which are ping-pong partners for H.

3.1. **Existence in connected groups.** The construction and study of free products in linear groups is a classical topic, going back way beyond Tits' celebrated work [74] establishing existence of free subgroups in linear groups which are not virtually solvable. Given a subset F of a linear group G, the existence of *simultaneous* ping-pong partners for elements of F (that is, elements which are ping-pong partners for every $h \in F$) has also been studied, see namely the works of Poznansky [60, Theorem 6.5] and Soifer & Vishkautsan [69, Theorem 1.3]. We also mention in passing the following open question asked by de la Harpe, cases of which are answered in the two works just cited.

Question 3.1 ([16, Question 16])**.** Let G be a connected semisimple real Lie group without compact factors, and let Γ be a Zariski-dense subgroup of the adjoint group $\mathrm{Ad}(G)$. Let F be a finite set of non-trivial elements of Γ. Does there exists an element $\gamma \in \Gamma$ of infinite order such that $\langle h, \gamma\rangle \cong \langle h\rangle * \langle \gamma\rangle$ for every $h \in F$?

Of course, if F is a subgroup, the condition that $\langle h, \gamma\rangle$ be freely generated for every element $h \in F$ does not imply that the subgroup $\langle F, \gamma\rangle$ is freely generated by F and γ. For instance, if for every $h \in F$ the subgroup $\langle h, \gamma\rangle$ of G is freely generated, then so is the subgroup $\langle (h_1, h_2), (\gamma, \gamma)\rangle$ of $G \times G$ for any $(h_1, h_2) \in F \times F$, but $\langle F \times F, (\gamma, \gamma)\rangle$ is not freely generated, as $(\gamma h_1 \gamma^{-1}, 1)$ commutes with $(1, h_2)$.

For this reason and others, we cannot directly use the works mentioned above; but we will use similar techniques to prove the following.

Theorem 3.2. *Let $\mathbf{G}$ be a connected algebraic F-group with center $\mathbf{Z}$. Let Γ be a Zariski-connected subgroup of $\mathbf{G}(F)$. Let $(H_i)_{i\in I}$ be a finite collection of finite subgroups of $\mathbf{G}(F)$, and set $C_i = H_i \cap \mathbf{Z}(F)$. Assume that for each $i \in I$ there exists a local field K_i containing F and a projective K_i-representation $\rho_i : \mathbf{G} \to \mathrm{PGL}_{V_i}$, where V_i is a finite-dimensional module over a finite division K_i-algebra D_i, with the following properties:*

- (Proximality) *$\rho_i(\Gamma)$ contains a proximal element;*
- (Transversality) *For every $h \in H_i \backslash C_i$ and every $p \in \mathbf{P}(V_i)$, the span of the set $\{\rho_i(xhx^{-1})p \mid x \in \Gamma\}$ is the whole of $\mathbf{P}(V_i)$.*

Let S be the collection of regular semisimple elements $\gamma \in \Gamma$ of infinite order, such that for all $i \in I$, the canonical map

$$(\langle \gamma\rangle \times C_i) *_{C_i} H_i \to \langle \gamma, H_i\rangle \leq \mathbf{G}(F)$$

is an isomorphism. Then S is dense in Γ for the join of the profinite topology and the Zariski topology.

Remark 3.3. The conclusion of the theorem amounts to the kernel of the canonical map

$$\langle \gamma\rangle * H_i \to \langle \gamma, H_i\rangle \leq \mathbf{G}(F)$$

being $\langle\langle [\gamma, C_i]\rangle\rangle$. Note that when $\mathbf{Z}(F)$ is trivial, the theorem states that for any $\gamma \in S$ and for all $i \in I$, the subgroup $\langle \gamma, H_i\rangle$ is freely generated by γ and H_i.

[1] In this paper, all reductive (in particular, all semisimple) algebraic groups are connected by definition. This convention sometimes differs in the literature. We also call *simple* a non-commutative algebraic group whose proper normal subgroups are finite (sometimes called 'almost simple' in the literature).

Remark 3.4. Note that the transversality condition implies that every ρ_i is irreducible. Moreover, the transversality condition holds equivalently for Γ or for its Zariski closure (it is a *Zariski-closed* condition). Thus, if Γ happens to be Zariski-dense (as is most common), this condition can be replaced by the analogue for $\mathbf{G}(K_i)$:

(Transversality′) *For every* $h \in H_i \backslash C_i$ *and every* $p \in \mathbf{P}(V_i)$*, the span of the set* $\{\rho_i(xhx^{-1})p \mid x \in \mathbf{G}(K_i)\}$ *is the whole of* $\mathbf{P}(V_i)$*.*

Remark 3.5. Theorem 3.2 is only meaningful for pseudo-reductive groups. Indeed, the F-unipotent radical $\mathrm{R}_{\mathrm{u},F}(\mathbf{G})$ must acts trivially under ρ_i, as the fixed-point set of $\mathrm{R}_{\mathrm{u},F}(\mathbf{G})$ is non-empty by the Lie–Kolchin theorem, hence is the whole of V_i. Thus each ρ_i factors through the pseudo-reductive quotient $\mathbf{G}/\mathrm{R}_{\mathrm{u},F}(\mathbf{G})$ of $\mathbf{G}$. We remind the reader that if $\operatorname{char} F = 0$, the full unipotent radical $\mathrm{R}_{\mathrm{u}}(\mathbf{G})$ of $\mathbf{G}$ is defined over F, hence pseudo-reductive groups are reductive (the converse always holding).

In subsequent sections, we will mostly be concerned with number fields and their archimedean completions, leaving aside the usual complications arising in positive characteristic.

Remark 3.6. There is no obvious analogue of Theorem 3.2 for HNN extensions. Indeed, $\mathbf{G}(F)$ may admit finite subgroups H containing a proper subgroup H_1 whose centralizer in $\mathbf{G}(F)$ is trivial. For instance, $\mathrm{PGL}_2(\mathbb{C})$ contains a copy of the symmetric group on 4 letters, whose alternating subgroup has trivial centralizer (see for instance [5, Proposition 1.1]). In such a situation, there is no HNN extension in $\mathbf{G}(F)$ of H with respect to the identity $H_1 \to H_1$, as any $g \in \mathbf{G}(F)$ centralizing H_1 is trivial, but $H *_{H_1}$ is not.

3.2. **Proximal dynamics in projective spaces.** Before proving Theorem 3.2, we need to extend a few known facts about the dynamics of the action of $\mathrm{GL}(V)$ on $\mathbf{P}(V)$ to projective spaces over division algebras. Foremost, we will need the contents of [74, §3] over a division algebra, but the proofs given by Tits are valid with minor adaptations to keep track of the D-structure and the fact D is not necessarily commutative. All of this is straightforward, so we will not rewrite arguments whenever they apply in the same way.

In this subsection, let K be a local field, D a division algebra of dimension d over K, and V a finite-dimensional right D-module. Recall that the absolute value $|\cdot|$ of K extends uniquely to an absolute value on D which will also denote by $|\cdot|$; we have the formula $|x| = |\mathrm{N}(x)|^{1/d}$ for $x \in D$.

With little deviation, we will follow the notations and conventions of [73] and [74], which the reader may consult along with [8] for background material on the representation theory of algebraic groups (including over division algebras).

Recall that GL_V is the algebraic K-group of automorphisms of the D-module V, so that for any F-algebra A, the group $\mathrm{GL}_V(A)$ is the group of automorphisms of the right $(D \otimes_K A)$-module $V \otimes_K A$. Provided $\dim V \geq 2$, the K-group PGL_V is the quotient of GL_V by its center (which is the multiplicative group of the center of D). The *projective general linear group* PGL_V acts on the *projective space* $\mathbf{P}(V)$ of V, which is the space of right D-submodules of V of dimension 1. The D-submodules of V and their images in $\mathbf{P}(V)$ are both called *(D-linear) subspaces*. A projective representation $\rho : \mathbf{G} \to \mathrm{PGL}_V$ of a K-group $\mathbf{G}$ is called *irreducible* if there are no proper non-trivial linear subspaces of $\mathbf{P}(V)$ stable under $\rho(\mathbf{G})$. A representation $\mathbf{G} \to \mathrm{GL}_V$ is then irreducible if and only if its projectivization is.

Given two subspaces X, Y of $\mathbf{P}(V)$, we denote their span by $X \vee Y$. If $X \cap Y = \emptyset$ and $X \vee Y = \mathbf{P}(V)$, we denote by $\operatorname{proj}(X, Y)$ the mapping $\pi : X \to Y$ defined by $\{\pi(p)\} = (X \vee \{p\}) \cap Y$. We will denote by $\mathring{C}$ the interior (for the local topology) of a subset C of $\mathbf{P}(V)$.

When it is needed to view V as a K-module instead of a D-module, we will add the corresponding subscript to the notation.

Definition 3.7. Let g be an element of $\mathrm{GL}_V(K)$ or $\mathrm{PGL}_V(K)$.

(1) Momentarily view V as a vector K-space, so as to identify GL_V with the subgroup of $\mathrm{GL}_{V,K}$ centralizing the right action of D on V, and likewise for PGL_V. The *attracting subspace of* g is the subspace $\mathrm{A}(g)$ of V which is the direct sum of the generalized eigenspaces (over some algebraic closure) associated to the eigenvalues of maximal absolute value of (any lift to GL_V of) g. The complementary set $\mathrm{A}'(g)$ is defined to be the direct sum of the remaining generalized eigenspaces of g. By construction, $V = \mathrm{A}(g) \oplus \mathrm{A}'(g)$.

Note that since the Galois group of any extension of K preserves the absolute value, it permutes the generalized eigenspaces of maximal absolute value, hence $\mathrm{A}(g)$ and $\mathrm{A}'(g)$ are stable under the Galois group and are indeed defined over K. Moreover, if g commutes with the action of D, then D preserves the generalized eigenspaces of g (after perhaps extending scalars). In this case, $\mathrm{A}(g)$ and $\mathrm{A}'(g)$ are themselves stable under D, i.e. they are D-subspaces of V.

The subspaces $\mathrm{A}(g)$ and $\mathrm{A}'(g)$ only depend on the image of g in PGL_V. In what follows, we will often omit projectivization from the notation as long as it causes no confusion between V and $\mathbf{P}(V)$.

(2) We call g *proximal* if $\dim_D \mathrm{A}(g) = 1$, in other words if $\mathrm{A}(g)$ is a point in $\mathbf{P}(V)$. In case $D = K$, this means that g has a unique eigenvalue (counting with multiplicity) of maximal absolute value. In general, this means that g has d (possibly different) eigenvalues of maximal absolute value. If both $\mathrm{A}(g)$ and $\mathrm{A}(g^{-1})$ are one-dimensional, we call g *biproximal*[2]. We call a (projective) representation $\rho : \Gamma \to (\mathrm{P})\mathrm{GL}_V(K)$ *proximal* if $\rho(\Gamma)$ contains a proximal element.

Proximal elements have contractive dynamics on $\mathbf{P}(V)$: if g is proximal, then for any $p \in \mathbf{P}(V) \setminus \mathrm{A}'(g)$ the sequence $(g^n \cdot p)_{n\in\mathbb{N}}$ converges to the point $\mathrm{A}(g)$ (see Lemma 3.8).

The complement $\mathbf{P}(V) \setminus X$ of a hyperplane $X \subset \mathbf{P}(V)$ can be identified with an affine space over D by choosing for V a system of coordinate functions $\xi = (\xi_0, \dots, \xi_{\dim \mathbf{P}(V)})$, $\xi_i \in V^*$, such that $X = \ker \xi_0$. The functions $\xi_i \xi_0^{-1}$ $(i = 1, \dots, \dim \mathbf{P}(V))$ then define affine coordinates on $\mathbf{P}(V) \setminus X$. If $g \in \mathrm{PGL}_V(K)$ stabilizes X, its restriction to $\mathbf{P}(V) \setminus X$ need not be an affine map in these coordinates, but will be semiaffine (with respect to conjugation by the factor by which g scales ξ_0). In particular, if $\mathbf{P}(V) \setminus X$ is seen as an affine space over K, then the restriction of g is K-affine.

For the rest of this section, we fix an *admissible* distance d on $\mathbf{P}(V)$, that is, a distance function $d : \mathbf{P}(V) \times \mathbf{P}(V) \to \mathbf{P}(V)$ inducing the local topology on $\mathbf{P}(V)$ and satisfying the property that for every compact subset C contained in an affine subspace of $\mathbf{P}(V)$, there exist constants $M, M' \in \mathbb{R}$ such that

$$M \cdot d_\xi|_{C\times C} \leq d|_{C\times C} \leq M' \cdot d_\xi|_{C\times C}.$$

Here d_ξ is the supremum distance with respect to the affine coordinates $(\xi_i \xi_0^{-1})_{i=1}^{\dim \mathbf{P}(V)}$ described above. Note that two different coordinate systems on the same affine subspace A of $\mathbf{P}(V)$ define comparable distance functions on this affine subspace. Moreover, if instead of using D-coordinates one views A as an affine K-space, the supremum distance in any set of affine K-coordinates will again be comparable to d_ξ.

As indicated by Tits, when K is an archimedean local field, any elliptic metric on $\mathbf{P}(V)$ is admissible. Tits also indicates in [74, §3.3] how to construct an admissible metric in the non-archimedean case by patching together different d_ξ's; this construction works identically over a division algebra.

[2]Biproximal elements are sometimes called 'very proximal' in the literature.

Having fixed an (admissible) distance d on $\mathbf{P}(V)$, the *norm* of a mapping $f : X \to \mathbf{P}(V)$ defined on some subset $X \subset \mathbf{P}(V)$ is the quantity

$$\|f\| = \sup_{\substack{p,q \in X \\ p \neq q}} \frac{d(f(p), f(q))}{d(p,q)}.$$

Note that the norm is submultiplicative: given mappings $f : X \to \mathbf{P}(V)$ and $g : Y \to X$, we have $\|f \circ g\| \leq \|f\| \cdot \|g\|$. Projective transformations always have finite norm [74, Lemma 3.5]. Indeed, given $g \in \mathrm{PGL}_V(K)$, the distance function d^g defined by $d^g(p,q) = d(gp, gq)$ is again admissible. Since $\mathbf{P}(V)$ is compact, it can be covered by finitely many compact sets contained in affine subspaces, on which the ratio between d^g and d is uniformly bounded, by admissibility.

We can now state the needed results from [74, §3] in our setting. The following two lemmas describe the dynamics of D-linear transformations.

Lemma 3.8 (Lemma 3.8 in [74])**.** *Let $g \in \mathrm{PGL}_V(K)$, let C be a compact subset of $\mathbf{P}(V)$ and let $r \in \mathbb{R}_{>0}$.*

- (i) *Suppose that g is proximal and that $C \cap \mathrm{A}'(g) = \emptyset$. Then there exists an integer N such that $\|g^n|_C\| < r$ for all $n > N$; and for any neighborhood U of $\mathrm{A}(g)$, there exists an integer N' such that $g^n C \subset U$ for all $n > N'$.*
- (ii) *Assume that, for some $m \in \mathbb{N}$, one has $\|g^m|_C\| < 1$ and $g^m C \subset \mathring{C}$. Then $\mathrm{A}(g)$ is a point contained in $\mathring{C}$.*

Note that in loc. cit. Tits assumes the existence of a semisimple proximal element; but as he indicates in the footnotes, this assumption is superfluous and the proof of the lemma is identical with an arbitrary proximal element.

Proof. The argument given by Tits applies, taking into account the following adaptations.

In part (i), the transformation g restricted to $\mathbf{P}(V) \setminus A'(g)$ is not necessarily D-linear, as was already mentioned. It is nevertheless K-linear, with eigenvalues of absolute value strictly smaller than 1 by assumption. So one can apply [74, Lemma 3.7 (i)] over K and use that the norms defined over D or K are comparable to conclude.

In part (ii), one cannot pick a representative of g in GL_V whose eigenvalues corresponding to the fixed point $p \in \mathbf{P}(V)$ equal one (as g may have different eigenvalues on the D-line p). Nevertheless, they are all of the same absolute value, which we can assume to be 1. If there is another eigenvalue of the same absolute value (i.e. if $\mathrm{A}(g) \neq \{p\}$), then the restriction of g to $\mathrm{A}(g)$ is a block-upper-triangular matrix in a well-chosen basis. Since the compact set C has non-empty interior, this contradicts the hypothesis of (ii). ☐

Lemma 3.9 (Lemma 3.9 in [74])**.** *Let $g \in \mathrm{PGL}_V(K)$ be semisimple, let $\bar{g} \in \mathrm{GL}_V(K)$ be a representative of g, let Ω be the set of eigenvalues of $\bar{g}$ (over an appropriate field extension of K) whose absolute value is maximum, let C be a compact subset of $\mathbf{P}(V) \setminus \mathrm{A}'(g)$, set $\pi = \mathrm{proj}(\mathrm{A}'(g), \mathrm{A}(g))$, and let U be a neighborhood of $\pi(C)$ in $\mathbf{P}(V)$.*

- (i) *There exists an infinite set $N \subset \mathbb{N}$ such that $\lim_{\substack{n \in N \\ n \to \infty}} (\lambda^{-1}\mu)^n = 1$ for all $\lambda, \mu \in \Omega$.*
- (ii) *The set $\{\|g^n|_C\| \mid n \in \mathbb{N}\}$ is bounded.*
- (iii) *If N is as in (i), $g^n C \subset U$ for almost all $n \in N$.*

Proof. The easiest way to obtain this lemma over the division algebra D is to take a representative of g in GL_V, see it as an K-linear transformation in $\mathrm{GL}_{V,K}$ and apply Tits' original lemma [74, Lemma 3.9]. Part (i) is then immediate.

For part (ii) and (iii), denote $\mathbf{P}_K(V)$ the projective space of V seen as a vector K-space. Since the canonical GL_V-equivariant map $q : \mathbf{P}_K(V) \to \mathbf{P}(V)$ is proper and continuous,

$C' = q^{-1}(C)$ is compact, and $U' = q^{-1}(U)$ is open. Thus [74, Lemma 3.9] applies with C' and U' over K, and in turn yields the same conclusions over D, since the norms of g restricted to C and C' bound each-other. □

We will also make use of a version of part (i) of Lemma 3.9 for multiple representations, due to Margulis and Soifer. They initially stated it for multiple vector spaces over the same local field, but as already observed in [60, Lemma 3.1], the proof is identical.

Lemma 3.10 (Lemma 3 in [55])**.** *Let $\{K_i\}_{i\in I}$, be a finite collection of local fields and V_i be a finite-dimensional vector K_i-space. Let g_i be a semisimple element of $\mathrm{GL}_{V_i}(K)$, and let $\Omega(g_i)$ be the set of eigenvalues of g_i whose absolute value is maximum. There exists an infinite subset $N \subset \mathbb{N}$ such that $\lim_{\substack{n\in N\\ n\to\infty}} (\lambda^{-1}\mu)^n = 1$ for all $i \in I$ and $\lambda, \mu \in \Omega(g_i)$.*

We are now ready to prove the following slight generalization of [60, Corollary 3.7], which is itself a refinement of both [74, Proposition 3.11] and [55, Lemma 8]. This proposition is a crucial piece of the proof of Theorem 3.2: it will be used to find enough biproximal elements in Γ.

Proposition 3.11 (Abundance of simultaneously biproximal elements)**.** *Let $\mathbf{G}$ be a connected algebraic F-group and let Γ be a Zariski-dense subgroup of $\mathbf{G}(F)$. Let $\{K_i\}_{i\in I}$ be a finite collection of local fields each containing F. For each $i \in I$, let $\rho_i : \mathbf{G} \to \mathrm{PGL}_{V_i}$ be an irreducible projective K_i-representation, where V_i is a finite-dimensional module over a finite division K_i-algebra D_i.*

Suppose that for each $i \in I$, $\rho_i(\Gamma)$ contains a proximal element. Then the set of regular semisimple elements $\gamma \in \Gamma$ such that $\rho_i(\gamma)$ is biproximal for every $i \in I$, is dense in Γ for the join of the Zariski topology and the profinite topology.

Proof. We follow the line of arguments given in [74, 55, 60], keeping track of the different representations, and using the extension of Tits' work to projective representations over a division algebra laid out above.

Given an arbitrary element $g \in \mathbf{G}(F)$, let us abbreviate $\rho_i(g)$ by g_i.

Step 1: The set of simultaneously proximal elements in Γ is Zariski-dense if it is non-empty.

Let $g \in \Gamma$ be such that g_i is proximal for all $i \in I$. Since ρ_i is irreducible, for each $i \in I$ the set of elements x of $\mathbf{G}(F)$ such that $x_i\mathrm{A}(g_i) \notin \mathrm{A}'(g_i)$ is non-empty and Zariski-open. Because $\mathbf{G}$ is Zariski-connected, the intersection of these sets remains non-empty (and Zariski-open). Let us then pick $x \in \Gamma$ satisfying $x_i\mathrm{A}(g_i) \notin \mathrm{A}'(g_i)$ for every $i \in I$.

By construction of x, we can pick a compact neighborhood C_i of $\mathrm{A}(g_i)$ in $\mathbf{P}(V_i)$ such that x_iC_i is disjoint from $A'(g)$. Since projective transformations have finite norm, we have $\max_{i\in I} \|x_i|_{C_i}\| < r$ for some $r \in \mathbb{R}$. By Lemma 3.8 (i), for each $i \in I$ there exists an integer N_i such that

$$\|g_i^n|_{x_iC_i}\| < r^{-1} \qquad \text{and} \qquad g_i^n(x_iC_i) \subset \mathring{C}_i \qquad \text{for } n > N_i.$$

Set $N_x = \max_{i\in I} N_i$. Then for any $i \in I$, we have that

$$\|g_i^n x_i|_{C_i}\| < 1 \qquad \text{and} \qquad (g_i^n x_i)C_i \subset \mathring{C}_i \qquad \text{for } n > N_x.$$

We deduce from Lemma 3.8 (ii) that $g_i^n x_i = \rho_i(gx)$ is proximal for every $n > N_x$.

Observe that the Zariski closure Z of $\{g^n \mid n > N_x\}$ in Γ has the property that $gZ \subset Z$. Since the Zariski topology is Noetherian, we deduce that $g^{m+1}Z = g^m Z$ for some $m \in \mathbb{N}$. This implies that $g^n Z = Z$ for every $n \in \mathbb{Z}$, and in particular that $g \in Z$. Let now $\overline{S}$ denote the Zariski closure in Γ of the set S of elements of Γ which are proximal under every ρ_i. We have shown that S contains $g^n x$ for each $x \in \Gamma$ chosen as above and $n > N_x$.

By our last observation, $\overline{S}x^{-1}$ contains g, hence $gx \in \overline{S}$. As this holds for every x in a Zariski-dense (open) subset of Γ, we conclude that $\overline{S}$ contains $g\Gamma = \Gamma$, as claimed.

Step 2: Γ contains a semisimple element that is simultaneously proximal.

We argue by induction on $\#I$. Fix $j \in I$, and suppose that there are elements $g, h \in \Gamma$ such that $\rho_j(h)$ is proximal and $\rho_i(g)$ is proximal for $i \in I \setminus \{j\}$. By Step 1, we may in addition assume that g and h are semisimple. Write $\pi_i = \mathrm{proj}(\mathrm{A}'(h_i), \mathrm{A}(h_i))$ for $i \neq j$, and $\pi_j = \mathrm{proj}(\mathrm{A}'(g_j), \mathrm{A}(g_j))$.

Let $N \subset \mathbb{N}$ be an infinite set such as afforded by Lemma 3.10 applied to the elements h_i for $i \neq j$ and g_j for $i = j$, so that we have $\lim_{\substack{n\in N\\ n\to\infty}} (\lambda^{-1}\mu)^n = 1$ for $\lambda, \mu \in \Omega(h_i)$ if $i \neq j$, and for $\lambda, \mu \in \Omega(g_j)$.

Since ρ_i is irreducible and Γ is Zariski-dense, as before we can fix $x \in \Gamma$ such that

$$x_i \mathrm{A}(g_i) \not\subset \mathrm{A}'(h_i) \quad \text{for every } i \in I.$$

Similarly, the elements $y \in \mathbf{G}(F)$ satisfying

$$\begin{aligned} y_i \cdot \pi_i(x_i \mathrm{A}(g_i)) &\notin \mathrm{A}'(g_i) \quad \text{for } i \in I \setminus \{j\}, \\ \text{and} \quad y_j \mathrm{A}(h_j) &\notin \left(x_j^{-1}\mathrm{A}'(h_j) \cap \mathrm{A}(g_j)\right) \vee \mathrm{A}'(g_j), \end{aligned}$$

form a non-empty Zariski-open subset of $\mathbf{G}(F)$. Let us then fix y such an element in Γ.

For $i \neq j$, let B_i be a compact neighborhood of $y_i \cdot \pi_i(x_i \mathrm{A}(g_i))$ disjoint from $\mathrm{A}'(g_i)$, and let B_j be a compact neighborhood of $x_j \cdot \pi_j(y_j \mathrm{A}(h_j))$ disjoint from $\mathrm{A}'(h_j)$. The latter exists because $\pi_j^{-1}(x_j^{-1}\mathrm{A}'(h_j)) \subset (x_j^{-1}\mathrm{A}'(h_j) \cap \mathrm{A}(g_j)) \vee \mathrm{A}'(g_j)$ does not contain $y_j \mathrm{A}(h_j)$. We also choose for $i \neq j$ a compact neighborhood C_i of $\mathrm{A}(g_i)$ disjoint from $x_i^{-1}\mathrm{A}'(h_i)$ and small enough to satisfy $y_i \cdot \pi_i(x_i C_i) \subset \mathring{B}_i$; and choose a compact neighborhood C_j of $\mathrm{A}(h_j)$ disjoint from $y_j^{-1}\mathrm{A}'(g_j)$ and satisfying $x_j \cdot \pi_j(y_j C_j) \subset \mathring{B}_j$.

The careful choice of B_i, C_i and N sets us up for the following applications of Lemmas 3.8 and 3.9. By Lemma 3.9, for each $i \neq j$ there exists $r_i \in \mathbb{R}$ and $N_i \in \mathbb{N}$ such that

$$\|h_i^n|_{x_i C_i}\| < r_i \text{ for } n \in \mathbb{N} \quad \text{and} \qquad y_i h_i^n x_i C_i \subset \mathring{B}_i \quad \text{for } n \in N,\ n > N_i.$$

Similarly, there exists $N_j \in \mathbb{N}$ and $r_j \in \mathbb{R}$ such that

$$\|g_j^n|_{y_j C_j}\| < r_j \text{ for } n \in \mathbb{N} \quad \text{and} \qquad x_j g_j^n y_j C_j \subset \mathring{B}_j \quad \text{for } n \in N,\ n > N_j.$$

By Lemma 3.8 (i), for each $i \neq j$ there exists $N_i' \in \mathbb{N}$ such that

$$\|g_i^n|_{B_i}\| < \left(\|y_i|_{y_i^{-1}B_i}\| \cdot r_i \cdot \|x_i|_{C_i}\|\right)^{-1} \quad \text{and} \qquad g_i^n B_i \subset \mathring{C}_i \quad \text{for } n > N_i'.$$

Similarly, there exists $N_j' \in \mathbb{N}$ such that

$$\|h_j^n|_{B_j}\| < \left(\|x_j|_{x_j^{-1}B_j}\| \cdot r_j \cdot \|y_j|_{C_j}\|\right)^{-1} \quad \text{and} \qquad h_j^n B_j \subset \mathring{C}_j \quad \text{for } n > N_j'.$$

Set $N' = \{n \in N \mid n > N_i \text{ and } n > N_i' \text{ for all } i \in I\}$. For $i \neq j$, we have by construction that

$$\|g_i^m y_i h_i^n x_i|_{C_i}\| < 1 \quad \text{and} \quad g_i^m y_i h_i^n x_i C_i \subset \mathring{C}_i \quad \text{for } m, n \in N'.$$

Similarly, we have that

$$\|h_j^n x_j g_j^m y_j|_{C_j}\| < 1 \quad \text{and} \quad h_j^n x_j g_j^m y_j C_j \subset \mathring{C}_j \quad \text{for } m, n \in N'.$$

We conclude from Lemma 3.8 (ii) that for all $m, n \in N'$, the element $g_i^m y_i h_i^n x_i$ is proximal for $i \neq j$, and so is $h_j^n x_j g_j^m y_j$. But $h_j^n x_j g_j^m y_j$ and $g_j^m y_j h_j^n x_j$ are conjugate, so $g^m y h^n x \in \Gamma$ is proximal under ρ_i for every $i \in I$.

In view of Step 1, the set of simultaneously proximal elements in Γ is Zariski-dense, so there is also a semisimple one as claimed.

Step 3: Γ contains an element which is simultaneously biproximal.

By Steps 1–2, there is a semisimple element $g \in \Gamma$ such that $\rho_i(g^{-1})$ is proximal for every $i \in I$. Let N be an infinite set such as afforded by Lemma 3.10. Replacing N by an appropriate subset, we may assume that the set $g^N = \{g^n \mid n \in N\}$ is Zariski-connected.

Since ρ_i is irreducible and Γ is Zariski-dense, the elements $x \in \mathbf{G}(F)$ such that

$$x_i \mathrm{A}(g_i) \not\subset \mathrm{A}'(g_i^{-1}) \quad \text{and} \quad x_i^{-1} \mathrm{A}(g_i) \not\subset \mathrm{A}'(g_i^{-1}) \quad \text{for every } i \in I$$

form a non-empty Zariski-open subset. Fix such an element $x \in \Gamma$. For the same reasons, the set U of elements $y \in \mathbf{G}(F)$ satisfying

$$\begin{aligned} & y_i \mathrm{A}(g_i^{-1}) \notin x_i \mathrm{A}'(g_i) \vee (x_i \mathrm{A}(g_i) \cap \mathrm{A}'(g_i^{-1})), \\ \text{and} \quad & y_i^{-1} x_i \mathrm{A}(g_i^{-1}) \notin \mathrm{A}'(g_i) \vee (\mathrm{A}(g_i) \cap x_i \mathrm{A}'(g_i^{-1})) \qquad \text{for every } i \in I \end{aligned}$$

is also non-empty and Zariski-open; fix $y \in U \cap \Gamma$.

Write $\pi_i = \mathrm{proj}(\mathrm{A}'(g_i), \mathrm{A}(g_i))$ and $\pi_i' = \mathrm{proj}(x_i \mathrm{A}'(g_i), x_i \mathrm{A}(g_i))$. For each $i \in I$, let B_i be a compact neighborhood of $\pi_i'(y_i \mathrm{A}(g_i^{-1}))$ disjoint from $\mathrm{A}'(g_i^{-1})$, and let B_i' be a compact neighborhood of $\pi_i(y_i^{-1} x_i \mathrm{A}(g_i^{-1}))$ disjoint from $x_i \mathrm{A}'(g_i^{-1})$. We also choose a compact neighborhood C_i of $\mathrm{A}(g_i^{-1})$ disjoint from $y_i^{-1} x_i \mathrm{A}'(g_i)$ satisfying $\pi_i'(y_i C_i) \subset \mathring{B}_i$, and a compact neighborhood C_i' of $y_i^{-1} x_i \mathrm{A}(g_i^{-1})$ disjoint from $\mathrm{A}'(g_i)$ satisfying $\pi_i(C_i') \subset \mathring{B}_i'$.

By Lemma 3.9 (ii), for each $i \in I$ there exist $N_i, N_i' \in \mathbb{N}$ and $r_i, r_i' \in \mathbb{R}$ such that

$$\begin{aligned} \|x_i g_i^n x_i^{-1}|_{y_i C_i}\| < r_i \text{ for } n \in \mathbb{N} \quad & \text{and} \quad & x_i g_i^n x_i^{-1} y_i C_i \subset \mathring{B}_i \quad & \text{for } n \in N,\ n > N_i, \\ \|g_i^n|_{C_i'}\| < r_i' \text{ for } n \in \mathbb{N} \quad & \text{and} \quad & g_i^n C_i' \subset \mathring{B}_i' \quad & \text{for } n \in N,\ n > N_i'. \end{aligned}$$

By Lemma 3.8 (i), for each $i \in I$ there exist $M_i, M_i' \in \mathbb{N}$ such that

$$\begin{aligned} \|g_i^{-n}|_{B_i}\| < \left(r_i \cdot \|y_i|_{C_i}\|\right)^{-1} \quad & \text{and} \quad & g_i^{-n} B_i \subset \mathring{C}_i \quad & \text{for } n > M_i, \\ \|x_i g_i^{-n} x_i^{-1}|_{B_i'}\| < \left(\|y_i^{-1}|_{y_i C_i'}\| \cdot r_i'\right)^{-1} \quad & \text{and} \quad & x_i g_i^{-n} x_i^{-1} B_i' \subset y_i \mathring{C}_i' \quad & \text{for } n > M_i'. \end{aligned}$$

Set $N_{x,y} = \{n \in N \mid n > \max \bigcup_{i \in I} \{N_i, N_i', M_i, M_i'\}\}$. We then have by construction that

$$\begin{aligned} \|g_i^{-n} x_i g_i^n x_i^{-1} y_i|_{C_i}\| < 1 \quad & \text{and} \quad & g_i^{-n} x_i g_i^n x_i^{-1} y_i C_i \subset \mathring{C}_i \quad & \text{for } n \in N_{x,y}, \\ \|y_i^{-1} x_i g_i^{-n} x_i^{-1} g_i^n|_{C_i'}\| < 1 \quad & \text{and} \quad & y_i^{-1} x_i g_i^{-n} x_i^{-1} g_i^n C_i' \subset \mathring{C}_i' \quad & \text{for } n \in N_{x,y}. \end{aligned}$$

We conclude from Lemma 3.8 (ii) that for all $n \in N_{x,y}$ and for each $i \in I$, the element $g^{-n} x g^n x^{-1} y$ is biproximal under ρ_i.

Step 4: The set of regular semisimple simultaneously biproximal elements is dense.

Let S denote the set of elements in Γ which are biproximal under every ρ_i. Let Λ be a normal subgroup of finite index in Γ, and let $\gamma \in \Gamma$. Because the set of regular semisimple elements is Zariski-open, it suffices to show that $S \cap \Lambda\gamma$ is Zariski-dense to prove the proposition.

Since Γ is Zariksi-connected and Λ has finite index in Γ, every coset of Λ is Zariski-dense. Moreover, if $h \in \Gamma$ is such that h_i is proximal, then $h^{|\Gamma:\Lambda|}$ is also proximal under ρ_i, and belongs to Λ. We can thus apply Steps 1–3 to Λ, to find an element $g \in \Lambda$ such that g_i is biproximal for every $i \in I$.

As before, the set U of elements $x \in \mathbf{G}(F)$ such that

$$x_i\gamma_i\mathrm{A}(g_i) \notin \mathrm{A}'(g_i) \quad \text{and} \quad \gamma_i^{-1}x_i^{-1}\mathrm{A}(g_i^{-1}) \notin \mathrm{A}'(g_i^{-1}) \quad \text{for every } i \in I$$

is Zariski-open and non-empty. In particular, $\Lambda \cap U$ is Zariski-dense in Γ; pick $x \in \Lambda \cap U$.

Let $C_i^{\pm}$ be a compact neighborhood of $\mathrm{A}(g_i^{\pm 1})$ such that $(x\gamma)_i^{\pm 1}C_i^{\pm}$ is disjoint from $\mathrm{A}'(g_i^{\pm 1})$. Since projective transformations have finite norm, we have that $\max_{i\in I} \|(x\gamma)_i^{\pm 1}|_{C_i^{\pm}}\| < r$ for some $r \in \mathbb{R}$. By Lemma 3.8 (i), there exist integers N_i^+ and N_i^- such that

$$\|g_i^n|_{x_i\gamma_iC_i^+}\| < r^{-1} \qquad \text{and} \qquad g_i^n x_i\gamma_i C_i^+ \subset \mathring{C}_i^+ \qquad \text{for } n > N_i^+.$$
$$\|g_i^{-n}|_{(x\gamma)_i^{-1}C_i^-}\| < r^{-1} \qquad \text{and} \qquad g_i^{-n}(x\gamma)_i^{-1}C_i^- \subset \mathring{C}_i^- \qquad \text{for } n > N_i^-.$$

For $N_x = \max \bigcup_{i\in I}\{N_i^+, N_i^-\}$, we then have for every $i \in I$ that

$$\|g_i^n x_i\gamma_i|_{C_i^+}\| < 1 \qquad \text{and} \qquad g_i^n x_i\gamma_i C_i^+ \subset \mathring{C}_i^+ \qquad \text{for } n > N_x.$$
$$\|g_i^{-n}\gamma_i^{-1}x_i^{-1}|_{C_i^-}\| < 1 \qquad \text{and} \qquad g_i^{-n}\gamma_i^{-1}x_i^{-1}C_i^- \subset \mathring{C}_i^- \qquad \text{for } n > N_x.$$

We deduce from Lemma 3.8 (ii) that $g_i^n x_i\gamma_i$ and $g_i^{-n}\gamma_i^{-1}x_i^{-1}$ are proximal for every $i \in I$ and for $n > N_x$. But $g_i^{-n}\gamma_i^{-1}x_i^{-1}$ and $\gamma_i^{-1}x_i^{-1}g_i^{-n}$ are conjugate, so $g_i^n x_i\gamma_i$ is in fact biproximal for every $i \in I$. Of course $g^n x\gamma \in \Lambda\gamma$, so we have shown that $S \cap \Lambda\gamma$ contains $g^n x\gamma$ for every $x \in \Lambda \cap U$ and $n > N_x$.

As was observed in Step 1, the Zariski closure of $\{g^n \mid n > N_x\}$ in Γ contains g. Thus the Zariski closure of $S \cap \Lambda\gamma$ contains $gx\gamma$ for every $x \in \Lambda \cap U$. As $\Lambda \cap U$ is Zariski-dense, so is $S \cap \Lambda\gamma$. This concludes the proof of the proposition. □

3.3. **Towards the proof of Theorem 3.2.** Before starting the proof of Theorem 3.2, we record the following lemmas.

Lemma 3.12. *Let K, D and V be as in §3.2. Let $\mathbf{G}$ be a connected K-subgroup of PGL_V, acting irreducibly on $\mathbf{P}(V)$. Suppose that $\mathbf{G}(K)$ contains a proximal element g_0. Then the set*

$$X = \{\mathrm{A}(g) \mid g \in \mathbf{G}(K) \text{ is proximal}\} \subseteq \mathbf{P}(V)$$

coincides with the orbit $\mathbf{G}(K) \cdot \mathrm{A}(g_0)$ and constitutes the unique irreducible projective subvariety of $\mathbf{P}(V)$ stable under $\mathbf{G}(K)$. In consequence, $\mathrm{Stab}_{\mathbf{G}}(\mathrm{A}(g_0))$ is a parabolic subgroup of $\mathbf{G}$.

Proof. By a theorem of Chevalley, there is a Zariski-closed $\mathbf{G}(K)$-orbit $Y \subseteq \mathbf{P}(V)$. Let $g \in \mathbf{G}(K)$ be proximal. Because $\mathbf{G}$ acts irreducibly on $\mathbf{P}(V)$, there exists $y \in Y \setminus \mathrm{A}'(g)$. We then have $g^n \cdot y \xrightarrow{n\to\infty} \mathrm{A}(g)$, thus $\mathrm{A}(g)$ lies in the closure of Y in the local hence in the Zariski topology. As Y was Zariski-closed, $\mathrm{A}(g) \in Y$. Since this happens for any proximal element g, we deduce that $X \subseteq Y$. As X is $\mathbf{G}(K)$-stable and Y is a single orbit, equality holds. It is now clear that X is the set of K-points of a projective variety $\mathbf{X}$, which is irreducible because $\mathbf{G}$ is.

Let $\mathbf{P} = \mathrm{Stab}_{\mathbf{G}}(\mathrm{A}(g))$ denote the stabilizer of $\mathrm{A}(g)$ in $\mathbf{G}$. The above shows that orbit map yields an isomorphism $\mathbf{G}/\mathbf{P} \to \mathbf{X}$, hence $\mathbf{G}/\mathbf{P}$ is a complete variety, meaning that $\mathbf{P}$ is parabolic. The same holds for every other proximal element. □

Remark 3.13. Lemma 3.12 can also be proven by arguing that if g_0 is proximal, $\mathrm{A}(g_0)$ must be a highest weight line.

Lemma 3.14 (Transversality)**.** *Let $\mathbf{G}$ be as in Lemma 3.12, and suppose that $\mathbf{G}(K)$ contains a proximal element g. For any $h \in \mathbf{G}(K)$, the set*

$$U_{h,g} = \{x \in \mathbf{G}(K) \mid xhx^{-1}\mathrm{A}(g) \notin \mathrm{A}'(g) \cup \mathrm{A}'(g^{-1})\}$$

is Zariski-open in $\mathbf{G}(K)$*. If* $h \in \mathbf{G}(K)$ *is such that the span of* $\{xhx^{-1}\mathrm{A}(g) \mid x \in \mathbf{G}(K)\}$ *is the whole of* $\mathbf{P}(V)$*, then* $U_{h,g}$ *is non-empty.*

Proof. The two sets

$$U_1 = \{x \in \mathbf{G}(K) \mid xhx^{-1}\mathrm{A}(g) \notin \mathrm{A}'(g)\}$$
$$U_2 = \{x \in \mathbf{G}(K) \mid xhx^{-1}\mathrm{A}(g) \notin \mathrm{A}'(g^{-1})\}$$

are Zariski-open by a standard argument: for any subspaces $W_1, W_2 \subseteq V$, the set $\{x \in \mathbf{G}(K) \mid x \cdot W_1 \subseteq W_2\}$ is Zariski-closed. We have to show they are both non-empty.

There is a minimal parabolic K-subgroup $\mathbf{B}$ of $\mathbf{G}$ that contains h. By Lemma 3.12, there is a conjugate $x\mathbf{B}x^{-1}$ of $\mathbf{B}$ which fixes $\mathrm{A}(g)$. But then for this choice of x, we surely have $xhx^{-1}\mathrm{A}(g) \notin \mathrm{A}'(g)$. This shows that U_1 is not empty.

Finally, U_2 is non-empty because of the assumption made on h. Indeed, U_2 being empty means $xhx^{-1}\mathrm{A}(g) \in \mathrm{A}'(g^{-1})$ for every $x \in \mathbf{G}(K)$, but the latter is a proper subspace of $\mathbf{P}(V)$. □

Remark 3.15. At first glance, Lemma 3.14 above may seem to be weaker than [60, Proposition 2.17]. Unfortunately, the proof of [60, Proposition 2.17] relies on [60, Proposition 2.11], whose statement is erroneous. The set of elements whose conjugacy class intersects a big Bruhat cell is in fact smaller than stated there (see for instance [33, Theorem 3.1] for a description in the case of SL_n). In consequence, the results of [60] are only valid under the additional assumption that the conjugacy classes of the elements h under consideration intersect a big Bruhat cell.

Proof of Theorem 3.2. For an arbitrary element $g \in \mathbf{G}(F)$, let us abbreviate $\rho_i(g)$ by g_i. For simplicity, we also write $H_i^* = H_i \setminus C_i$.

Fix a normal subgroup Λ of finite index in Γ, and fix $\gamma_0 \in \Gamma$. First, because of the proximality hypothesis, Proposition 3.11 applied to the Zariski-closure $\mathbf{H}$ of Γ in $\mathbf{G}$ states that the set S' of regular semisimple elements $\gamma' \in \Lambda\gamma_0$ such that $\rho_i(\gamma')$ is biproximal for every $i \in I$, is Zariski-dense in Γ. Pick $\gamma' \in S'$.

Second, using the transversality hypothesis on ρ_i, we exhibit a simultaneously biproximal element in $\Lambda\gamma_0$ acting transversely to every H_i. By Lemma 3.14, for every $i \in I$ and every $h \in H_i^*$ the sets

$$U_{i,h,\gamma'^{\pm 1}} = \{x \in \mathbf{H}(F) \mid x_i h_i x_i^{-1}\mathrm{A}(\gamma_i'^{\pm 1}) \notin \mathrm{A}'(\gamma_i') \cup \mathrm{A}'(\gamma_i'^{-1})\}$$

are Zariski-open and non-empty. In consequence, we can pick an element λ in the Zariski-dense set $\Lambda \cap U_{\gamma'}$, where $U_{\gamma'} = \bigcap_{i \in I}\bigcap_{h \in H_i^*}(U_{i,h,\gamma'} \cap U_{i,h,\gamma'^{-1}})$. Setting $\gamma = \lambda^{-1}\gamma'\lambda$, we see that $\gamma \in S'$, while for any $h \in H_i^*$,

$$h_i\mathrm{A}(\gamma_i) \notin \mathrm{A}'(\gamma_i) \cup \mathrm{A}'(\gamma_i{}^{-1}) \quad \text{and} \quad h_i\mathrm{A}(\gamma_i{}^{-1}) \notin \mathrm{A}'(\gamma_i) \cup \mathrm{A}'(\gamma_i{}^{-1}).$$

Next, we construct the sets that will allow us to apply Lemma 2.1. Given $i \in I$, let $P_i^{\pm}$ be a compact neighborhood of $\mathrm{A}(\gamma_i{}^{\pm 1})$ in $\mathbf{P}(V_i)$ small enough to achieve $(H_i^* \cdot P_i^{\pm}) \cap (\mathrm{A}'(\gamma_i) \cup \mathrm{A}'(\gamma_i{}^{-1})) = \emptyset$. Such a set exists by construction of γ: by local compactness, the complement of the closed set $H_i^* \cdot (\mathrm{A}'(\gamma_i) \cup \mathrm{A}'(\gamma_i{}^{-1}))$ contains a compact neighborhood of $\mathrm{A}(\gamma_i{}^{\pm 1})$. In the same way, we can arrange that also

$$(H_i^* \cdot P_i^{\pm}) \cap (P_i^+ \cup P_i^-) = \emptyset.$$

Note that $\mathbf{Z}(F)$ fixes $\mathrm{A}(\gamma_i)$ and $\mathrm{A}(\gamma_i{}^{-1})$. The finite intersection $\bigcap_{c \in C_i}(c \cdot P_i^{\pm})$ is thus again a compact neighborhood of $\mathrm{A}(\gamma_i{}^{\pm 1})$. Replacing $P_i^{\pm}$ by this intersection, we will further assume that $P_i^{\pm}$ is stable under C_i.

Set $P_i = P_i^+ \cup P_i^-$ and set

$$Q_i = H^* \cdot P_i;$$

these two subsets of $\mathbf{P}(V_i)$ are compact, disjoint, and preserved by C_i. As $Q_i \cap (\mathrm{A}'(\gamma_i) \cup \mathrm{A}'(\gamma_i{}^{-1})) = \emptyset$, Lemma 3.8 (i) shows that there exists $N \in \mathbb{N}$ such that for any $n > N$,

$$\gamma_i{}^n Q_i \subset P_i \quad \text{and} \quad \gamma_i{}^{-n} Q_i \subset P_i.$$

Pick $N_1 > N$ with $N_1 = 1 \mod |\Gamma : \Lambda|$, so that $\gamma^{N_1+n|\Gamma:\Lambda|} \in \Lambda\gamma_0$ for every $n \in \mathbb{Z}$.

For each $i \in I$, Lemma 2.1 now applies to the subgroups $\langle \gamma^{N_1+n|\Gamma:\Lambda|} \rangle \times C_i$ and H_i of $\mathbf{G}(F)$, with the sets P_i and Q_i constructed above. We conclude that for every $i \in I$ and all $n \in \mathbb{N}$, the subgroup $\langle \gamma^{N_1+n|\Gamma:\Lambda|}, H_i \rangle$ is the free amalgamated product $(\langle \gamma^{N_1+n|\Gamma:\Lambda|} \rangle \times C_i) *_{C_i} H_i$.

This establishes that $S \cap \Lambda\gamma_0$ contains $\gamma^{N_1+n|\Gamma:\Lambda|}$ for every $n \in \mathbb{N}$; it remains to show that $S \cap \Lambda\gamma_0$ is Zariski-dense.

The Zariski closure Z of $\{\gamma^{N_1+n|\Gamma:\Lambda|} \mid n \in \mathbb{N}\}$ satisfies $\gamma^{|\Gamma:\Lambda|} Z \subset Z$. Since the Zariski topology is Noetherian, it follows that $\gamma^{(m+1)|\Gamma:\Lambda|} Z = \gamma^{m|\Gamma:\Lambda|} Z$ for some $m \in \mathbb{N}$, and in turn that $\gamma \in Z$.

We have seen that S' is Zariski-dense, and that for each $\gamma' \in S'$, the set $\Lambda \cap U_{\gamma'}$ is Zariski-dense. In consequence, the set $S'' = \{(\gamma', \lambda) \in \Gamma \times \Gamma \mid \gamma' \in S', \lambda \in \Lambda \cap U_{\gamma'}\}$ is Zariski-dense in $\Gamma \times \Gamma$. Indeed, its closure contains $\overline{\{\gamma'\} \times S_{\gamma'}} = \{\gamma'\} \times \Gamma$ for each $\gamma' \in S'$, therefore contains $\overline{S' \times \{\gamma\}} = \Gamma \times \{\gamma\}$ for each $\gamma \in \Gamma$.

Since the conjugation map $\mathbf{H} \times \mathbf{H} \to \mathbf{H} : (x, y) \mapsto y^{-1}xy$ is dominant, it sends S'' to a Zariski-dense subset of Γ. Following the argument above, the Zariski closure of $S \cap \Lambda\gamma_0$ contains the image of S''. This proves the theorem. □

Remark 3.16. Each of the two properties assumed in Theorem 3.2 can be satisfied individually. Given a finitely generated Zariski-dense subgroup of a (connected) semisimple algebraic group, the existence of a local field and a representation satisfying the proximality property was first shown by Tits (see the proof of [74, Proposition 4.3]). A refinement to non-connected simple groups can also be found in [55, Theorem 1].

The second property, transversality, can be established for one given element $h \in H_i \setminus C_i$ using representation-theoretic techniques. But it is not always possible to find a representation that works for all $h \in H_i$ at the same time.

Even so, it may not always be possible to find a single representation which satisfies both properties of Theorem 3.2 simultaneously. Our next task will be to construct such a representation for real inner forms of SL_n and $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_n)$. This will be sufficient for the applications appearing in §4 & §5.

3.4. **Constructing a proximal and transverse representation for inner $\mathbb{R}$-forms of SL_n and GL_n.** Let D be a finite division $\mathbb{R}$-algebra and set $d = \dim_{\mathbb{R}} D$. Let $n \geq 2$ and let $\mathbf{H}$ be any algebraic $\mathbb{R}$-group in the isogeny class of SL_{D^n} or GL_{D^n}, viewing D^n as a right D-module. For example, if $D = \mathbb{C}$ this means that $\mathbf{H}$ is a quotient of the $\mathbb{R}$-group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_n)$ or $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{GL}_n)$ by a (finite) central subgroup. The *standard projective representation* of $\mathbf{H}$ is the canonical morphism $\rho_{\mathrm{st}} : \mathbf{H} \to \mathrm{PGL}_{D^n}$. This is the projective representation which will exhibit both proximal and transverse elements.

First, we recall that an element $g \in \mathbf{G}(\mathbb{R})$, in some reductive $\mathbb{R}$-group $\mathbf{G}$, is called $\mathbb{R}$-*regular* if the number of eigenvalues (counted with multiplicity) of $\mathrm{Ad}(g)$ of absolute value 1 is minimal. Any $\mathbb{R}$-*regular* element is semisimple (see [62, Remark 1.6.1]), and when $\mathbf{G}$ is split, every $\mathbb{R}$-regular element is regular.

With $\mathbf{H}$ as specified above, an element $g \in \mathbf{H}(\mathbb{R})$ is $\mathbb{R}$-regular if and only if some (any) representative of $\rho_{\mathrm{st}}(g)$ in $\mathrm{GL}_{D^n}(\mathbb{R})$ is conjugate to a diagonal n-by-n matrix with entries in D of distinct absolute values. Indeed, if $\rho_{\mathrm{st}}(g)$ is represented by $\mathrm{diag}(a_1, \ldots, a_n)$ with $|a_i| \neq |a_j|$ for $i \neq j$, the absolute values of the eigenvalues of $\mathrm{Ad}(g)$ are $\{|a_i a_j^{-1}|\}_{1 \leq i,j \leq 1}$ (with the right multiplicities) and are equal to 1 only for $i = j$, which is the least possible occurrences. Conversely, if g is $\mathbb{R}$-regular, the centralizer of the $\mathbb{R}$-regular element $\rho_{\mathrm{st}}(g)$ contains a unique maximal $\mathbb{R}$-split torus $\mathbf{S}$ of PGL_{D^n} (see [62, Lemma 1.5]). Thus $\rho_{\mathrm{st}}(g)$ belongs to the centralizer of $\mathbf{S}(\mathbb{R})$, which, up to conjugation, is the subgroup of (classes of) diagonal n-by-n matrices with entries in D; say $\rho_{\mathrm{st}}(g)$ is represented by $\mathrm{diag}(a_1, \dots, a_n)$. The absolute values of the eigenvalues of $\mathrm{Ad}(g)$ are again $\{|a_i a_j^{-1}|\}_{1 \le i,j \le 1}$. From the $\mathbb{R}$-regularity of $\rho_{\mathrm{st}}(g)$, we deduce that each value $|a_i a_j^{-1}|$ with $i \neq j$ must differ from 1, as claimed.

It follows from this description that if $\ell_{\max}$ (resp. $\ell_{\min}$) denotes the D-line in D^n on which a $\mathbb{R}$-regular element $g \in \mathbf{H}(\mathbb{R})$ acts by multiplication by an element of $D^\times$ of largest (resp. smallest) absolute value, then $\ell_{\max} = \mathrm{A}(g)$ is the attracting subspace of g (resp. $\ell_{\min} = \mathrm{A}(g^{-1})$), so that g is biproximal.[3] We record this here.

Lemma 3.17. *Let $\mathbf{H}$ and ρ_{st} be as above. Any $\mathbb{R}$-regular element $g \in \mathbf{H}(\mathbb{R})$ is biproximal under ρ_{st}.*

So, in order to exhibit proximal elements in $\rho_{\mathrm{st}}(\Gamma)$ for $\Gamma \le \mathbf{H}(\mathbb{R})$ a Zariski-dense subgroup, it suffices to show Γ admits a $\mathbb{R}$-regular element. This is the content of the following theorem, due to Benoist and Labourie [6, A.1 Théorème]. We also refer the reader to the direct proof given by Prasad in [61].

Theorem 3.18 (Abundance of $\mathbb{R}$-regular elements, A.1 Théorème in [6])**.** *Let $\mathbf{G}$ be a reductive $\mathbb{R}$-group. Let Γ be a Zariski-dense subgroup of $\mathbf{G}(\mathbb{R})$. The subset of $\mathbb{R}$-regular elements in Γ is Zariski-dense.*

Corollary 3.19. *Let $\mathbf{H}$ and ρ_{st} be as above. Let Γ be a Zariski-dense subgroup of $\mathbf{H}(\mathbb{R})$. The elements $g \in \Gamma$ such that $\rho_{\mathrm{st}}(g)$ is biproximal, form a Zariski-dense subset of Γ.*

Remark 3.20. The existence of elements proximal under ρ_{st} in any Zariski-dense sub(semi)group can also be established using the results of Goldsheid and Margulis [26, Theorem 6.3] (see also [1, 3.12–14]). This approach is more tedious, as the standard representation of GL_{D^n} does not admit proximal elements if D^n is seen as a vector $\mathbb{R}$-space (which is in fact one of the motivations to extend the framework of [74] to division algebras). Instead, one should embed $\mathbf{P}_D(D^n)$ inside $\mathbf{P}_{\mathbb{R}}(\bigwedge_{\mathbb{R}}^d D^n)$ via the Plücker embedding, and exhibit proximal elements in that projective representation.

Next, we move on to the question of transversality. It turns out that under ρ_{st}, every non-central element $h \in \mathbf{H}(\mathbb{R})$ satisfies the transversality condition of Theorem 3.2.

Proposition 3.21. *Let $\mathbf{H}$ and ρ_{st} be as above. Let $h \in \mathbf{H}(\mathbb{R})$ be non-central. For every $p \in \mathbf{P}(D^n)$, the span of $\{\rho_{\mathrm{st}}(xhx^{-1})p \mid x \in \mathbf{H}(\mathbb{R})\}$ is the whole of $\mathbf{P}(D^n)$.*

Proof. Taking preimages in GL_{D^n}, we may without loss of generality work with the action of GL_{D^n} on D^n instead of $\rho_{\mathrm{st}}(\mathbf{H}) = \mathrm{PGL}_{D^n}$ on $\mathbf{P}(D^n)$. We will show in this setting that, for every non-zero $v \in D^n$ and every non-central $h \in \mathrm{GL}_{D^n}(\mathbb{R})$, the $\mathbb{R}$-span of $\{xhx^{-1} \cdot v \mid x \in \mathrm{SL}_{D^n}(\mathbb{R})\}$ is the whole of D^n. The statement of the proposition then follows immediately by projectivization.

Viewing $\mathrm{End}_D(D^n)$ as a vector $\mathbb{R}$-space, the conjugation action defines a linear representation of SL_{D^n} on $\mathrm{End}_D(D^n)$. This representation decomposes into two irreducible components: a copy of the trivial representation given by the action of SL_{D^n} on the center of $\mathrm{End}_D D^n$, and a copy of the adjoint representation given by the action of SL_{D^n} on the subspace $\mathfrak{sl}_n(D)$ of traceless endomorphisms.

When h is not central, it admits a distinct conjugate xhx^{-1} of the same trace, hence the $\mathbb{R}$-span W_h of $\{xhx^{-1} \mid x \in \mathrm{SL}_{D^n}(\mathbb{R})\}$ contains for some $g \in \mathrm{SL}_{D^n}(\mathbb{R})$ the nonzero traceless

[3] Conversely, there exists a representation under which any proximal element is $\mathbb{R}$-regular, see [62, Lemma 3.4].

element $h' = h - ghg^{-1}$. In turn, W_h contains the $\mathbb{R}$-span $W_{h'}$ of $\{xh'x^{-1} \mid x \in \mathrm{SL}_{D^n}(\mathbb{R})\}$, a SL_{D^n}-stable subspace of $\mathfrak{sl}_n(D)$ which must equal $\mathfrak{sl}_n(D)$, as the latter is irreducible for the adjoint action. Thus, either $W_h = \mathfrak{sl}_n(D)$ if $\mathrm{Tr}(h) = 0$, or $W_h = \mathrm{End}_D(D^n)$ if $\mathrm{Tr}(h) \neq 0$.

Finally, for any non-zero $v \in D^n$ we have that $\mathfrak{sl}_n(D) \cdot v = D^n$, from which we conclude that the $\mathbb{R}$-span of $\{xhx^{-1} \cdot v \mid x \in \mathrm{SL}_{D^n}(\mathbb{R})\}$ contains $W_h \cdot v = D^n$. □

Definition 3.22. Given a reductive F-group $\mathbf{G}$ with center $\mathbf{Z}$ and a subgroup $H \leq \mathbf{G}(F)$, for the purposes of this paper, we will say that H *almost embeds in a (simple) quotient* $\mathbf{Q}$ *of* $\mathbf{G}$ if there exists a (simple) quotient $\mathbf{Q}$ of $\mathbf{G}$ for which the kernel of the restriction $H \to \mathbf{Q}(F)$ is contained in $\mathbf{Z}(F)$.

It is clear that if $\mathbf{Q}$ is a simple factor of $\mathbf{G}$ and H is a subgroup of $\mathbf{Q}(F)$, then H almost embeds in $\mathbf{Q}$. In particular, if $\mathbf{G}$ is itself simple, every subgroup almost embeds in a simple quotient.

With this, we are ready to prove the following application of Theorem 3.2, establishing the abundance of simultaneous ping-pong partners for finite subgroups in products of inner forms of SL_n and GL_n which almost embed in a factor.

Theorem 3.23. *Let* $\mathbf{G}$ *be a reductive* $\mathbb{R}$*-group whose simple quotients are each isogenic to* PGL_{D^n} *for* D *some finite division* $\mathbb{R}$*-algebra and* $n \geq 2$*, and let* $\mathbf{Z}$ *denote its center. Let* Γ *be a subgroup of* $\mathbf{G}(\mathbb{R})$ *whose image in* $\mathrm{Ad}\,\mathbf{G}$ *is Zariski-dense. Let* $(H_i)_{i\in I}$ *be a finite collection of finite subgroups of* $\mathbf{G}(\mathbb{R})$*, and set* $C_i = H_i \cap \mathbf{Z}(\mathbb{R})$*.*

Suppose that for each $i \in I$*, there exists a simple quotient* $\mathbf{Q}_i$ *of* $\mathbf{G}$ *for which the kernel of the projection* $H_i \to \mathbf{Q}_i(\mathbb{R})$ *is contained in* C_i*. Then the collection of regular semisimple elements* $\gamma \in \Gamma$ *of infinite order such that for all* $i \in I$*, the canonical map*

$$(\langle \gamma \rangle \times C_i) *_{C_i} H_i \to \langle \gamma, H_i \rangle \leq \mathbf{G}(\mathbb{R})$$

is an isomorphism, is dense in Γ *for the join of the profinite topology and the Zariski topology.*

Proof. By assumption, every simple quotient of $\mathbf{G}$ admits as further quotient PGL_{D^n}, for D some finite division $\mathbb{R}$-algebra and $n \geq 2$. For $i \in I$, let ρ_i denote the composite of the quotient map $\mathbf{G} \to \mathbf{Q}_i$ with the standard projective representation $\mathbf{Q}_i \to \mathrm{PGL}_{D_i^{n_i}}$, where D_i, n_i are an appropriate division $\mathbb{R}$-algebra and integer. Note that ρ_i factorizes $\mathbf{G} \to \mathrm{Ad}\,\mathbf{G} \to \mathrm{PGL}_{D_i^{n_i}}$.

Corollary 3.19 shows that the set of elements in $\rho_i(\Gamma)$ which are biproximal is Zariski-dense in $\mathrm{PGL}_{D_i^{n_i}}$; a fortiori, $\rho_i(\Gamma)$ contains a proximal element. Moreover, since C_i is the kernel of $\rho_i : H_i \to \mathrm{PGL}_{D_i^{n_i}}(\mathbb{R})$ by construction, every $h \in H_i \setminus C_i$ maps to a non-central element under ρ_i. Proposition 3.21 then precisely states that ρ_i satisfies the transversality condition of Theorem 3.2. We are thus at liberty to apply Theorem 3.2 to $\Gamma \leq \mathbf{G}(\mathbb{R})$ and the collection $(H_i)_{i\in I}$ (see also Remark 3.4), deducing this theorem. □

Remark 3.24. Let F be any field, and let $\mathbf{G}$ be a reductive F-group with center $\mathbf{Z}$. In order for a subgroup $H \leq \mathbf{G}(F)$ to admit a ping-pong partner in $\mathbf{G}(F)$, it is necessary that H almost embeds in a simple factor. Indeed, if the subgroup $\langle \gamma, H \rangle$ is the free amalgamated product of $\langle \gamma \rangle \times C$ and H over $C = H \cap \mathbf{Z}(F)$, then in the quotient $\mathbf{G}/\mathbf{Z}$, the image of $\langle \gamma, H \rangle$ is certainly freely generated by the images of γ and H. But $\mathbf{G}/\mathbf{Z}$ is the direct product of adjoint simple quotients of $\mathbf{G}$, so by Proposition 2.7, H/C embeds in (the F-points of) one of these factors.

In other words, Theorem 3.23 states that the finite subgroups $(H_i)_{i\in I}$ under consideration admit simultaneous ping-pong partners in Γ *if and only if* each H_i almost embeds in a simple factor.

Remark 3.25. There are versions of Theorem 3.23 for semisimple $\mathbb{R}$-groups of other types, but proving them requires a more careful study of the representation theory of $\mathbf{G}$ to exhibit a representation playing the role of ρ_{st}. However, as indicated in Remark 3.16, there are cases where one needs additional information on the H_i to get a representation satisfying the transversality assumption of Theorem 3.2.

There are also versions for other local fields. However, to prove those one needs additional information on Γ. Indeed, over a local field different from $\mathbb{R}$, bounded Zariski-dense subgroups exist, and a bounded subgroup obviously never admits proximal elements.

4. Constructing amalgams between two given finite subgroups of products of $\mathrm{GL}_n(D)'$s

Conventions: throughout the remainder of this article G will denote a finite group. All orders will be understood to be $\mathbb{Z}$-orders. We also use the following notations:

- Whenever we say that a given algebra A is a finite algebra we mean that A is finite dimensional
- $\mathrm{PCI}(FG)$ for the set of primitive central idempotents of FG
- $\pi_e : \mathcal{U}(FG) \twoheadrightarrow FGe$ projection to a simple component
- $\mathrm{Emb}_G(H)$ is the set of $e \in \mathrm{PCI}(\mathbb{Q}G)$ with $H \cap \ker(\pi_e) = 1$ (see (6))

Theorem 3.23 gives a satisfactory existence result of ping-pong partners for finite subgroups in a direct product of groups of the form $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, or $\mathrm{GL}_n(\mathbb{H})$ with $n \geq 2$.

The end goal of this paper being the study of free amalgamated products with finite subgroups inside $\mathcal{U}(\mathcal{O})$, the unit group of an order $\mathcal{O}$ in a finite semisimple algebra A over a number field F, we record the following application of Theorem 3.23 to finite subgroups in $\mathcal{U}(\mathcal{O})$.

Corollary 4.1. *Let F be a number field, A be a finite semisimple F-algebra, and $\mathcal{O}$ be an order in A. Let Γ be a Zariski-dense subgroup of $\mathcal{U}(\mathcal{O})$. Let H be a finite subgroup of $\mathcal{U}(A)$, and C its intersection with the center of A.*

There exists $\gamma \in \Gamma$ of infinite order with the property that the canonical map

$$(\langle\gamma\rangle \times C) *_C H \to \langle\gamma, H\rangle$$

is an isomorphism, if and only if H almost embeds in Ae for some $e \in \mathrm{PCI}(A)$ for which Ae is neither a field nor a totally definite quaternion algebra.

Moreover, in the affirmative, the set of such elements γ is dense in the join of the Zariski and the profinite topology.

In particular, a free product $\mathbb{Z} * H$ exists in $\mathcal{U}(\mathcal{O})$ if and only if C is trivial and H embeds in a factor Ae which is neither a field nor a totally definite quaternion algebra.

Proof. By Wedderburn's theorem, every semisimple F-algebra A factors as

$$A = \mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_m),$$

for V_i an n_i-dimensional right module over some finite division F-algebra D_i, $i = 1, \ldots m$. In consequence, the F-group $\mathbf{G}$ of units of A is the reductive group

$$\mathrm{GL}_{D_1^{n_1}} \times \cdots \times \mathrm{GL}_{D_m^{n_m}} .$$

We can base-change $\mathbf{G}$ to the $\mathbb{R}$-group $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \times_{\mathbb{Q}} \mathbb{R}$, whose $\mathbb{R}$-points $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$ are a product of groups of the form $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, or $\mathrm{GL}_n(\mathbb{H})$, for various $n \geq 1$.

Any subgroup H of $\mathcal{U}(A) = \mathbf{G}(F)$ embeds in $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$. In fact, H almost embeds in a F-simple factor of $\mathbf{G}$ if and only if it does so in a $\mathbb{R}$-simple factor of $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \times_{\mathbb{Q}} \mathbb{R}$. More precisely, let $K_1, \ldots, K_s$ denote the summands of the étale $\mathbb{R}$-algebra $F \otimes_{\mathbb{Q}} \mathbb{R}$; they are precisely the different archimedean completions of F. Given a finite division algebra

D over F, let D_{ij} be the division $\mathbb{R}$-algebras such that $D \otimes_F K_i \cong \prod_{j=1}^{m_i} \mathrm{M}_{r_{ij}}(D_{ij})$ as $\mathbb{R}$-algebras. The group $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{D^n} \times_{\mathbb{Q}} \mathbb{R}$ then factors into the product $\prod_{i=1}^{s} \prod_{j=1}^{m_i} \mathrm{GL}_{D_{ij}^{nr_{ij}}}$. The image of $\mathrm{GL}_{D^n}(F)$ in this product is obtained by embedding it diagonally using the canonical maps $\mathrm{GL}_{D^n}(F) \to \mathrm{GL}_{D^n}(K_i) \to \mathrm{GL}_{D_{ij}^{nr_{ij}}}(\mathbb{R})$. Thus if H (almost) embeds in a factor (P)GL_{D^n} over F, then it does so in any of the (P)$\mathrm{GL}_{D_{ij}^{nr_{ij}}}$ over $\mathbb{R}$, and the converse is obvious.

Now, a simple quotient $\mathrm{PGL}_{D_{ij}^{nr_{ij}}}$ over $\mathbb{R}$ of a given factor GL_{D^n} of $\mathbf{G}$ satisfies $nr_{ij} = 1$, if and only if the jth factor in $Ae \otimes_F K_i$ is a division algebra, where e is the projection onto the factor of A corresponding to GL_{D^n}. In other words, the factor GL_{D^n} has a simple quotient $\mathrm{PGL}_{D_{ij}^{nr_{ij}}}$ with $nr_{ij} \geq 2$ for some i, j, if and only if Ae is not a division algebra which remains so under every archimedean completion of its center. This amounts in turn to Ae not being a field nor a totally definite quaternion algebra.

Next, let $\mathbf{G}_{\mathrm{is}}$ denote the $\mathbb{R}$-subgroup of $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \times_{\mathbb{Q}} \mathbb{R}$ which is the direct product of those subgroups $\mathrm{GL}_{D_{ij}^{nr_{ij}}}$ for which $nr_{ij} \geq 2$. Since $\mathcal{U}(\mathcal{O})$ is an arithmetic subgroup of $\mathcal{U}(A) = \mathbf{G}(F)$, a classical theorem of Borel and Harish-Chandra [7] attests that the connected component of $\mathcal{U}(\mathcal{O})$ in $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \times_{\mathbb{Q}} \mathbb{R}$ is a lattice in the derived subgroup $\mathcal{D}\mathbf{G}_{\mathrm{is}}$ of $\mathbf{G}_{\mathrm{is}}$. In consequence, the image of Γ in $\mathrm{Ad}\, \mathbf{G}_{\mathrm{is}}$ is Zariski-dense.

Let f denote the canonical map $\mathbf{G}(\mathbb{R}) \to \mathrm{Ad}\, \mathbf{G}_{\mathrm{is}}(\mathbb{R})$, whose kernel is the product of the compact factors of $\mathbf{G}(\mathbb{R})$ with its center. Note that $\ker f$ commutes with $\mathbf{G}_{\mathrm{is}}(\mathbb{R})$, and that $\ker f \cap \Gamma$ is finite.

In view of all the above, provided H satisfies the embedding condition, we deduce from Theorem 3.23 applied to $\mathrm{Ad}\, \mathbf{G}_{\mathrm{is}}$ the existence of a dense set $S \subset f(\Gamma)$ of ping-pong partners for $f(H)$. By Lemma 2.3, the preimage $f^{-1}(S) \cap \Gamma$ consists of elements $\gamma \in \Gamma$ for which the canonical map $(\langle \gamma \rangle \times C) *_C H \to \langle \gamma, H \rangle$ is an isomorphism.

As S is dense in the join of the Zariski and the profinite topology, the same holds for $f^{-1}(S) \cap \Gamma$. Indeed, if $\Lambda\gamma_0$ is a coset of finite index in Γ, and U is a Zariski-open subset of Γ intersecting it, perhaps after shrinking and translating by $\ker f \cap \Gamma$, we can arrange that $\Lambda\gamma_0$ and U are contained in the connected component Γ° of Γ, and that $(\ker f \cap \Gamma^\circ) \cdot U = U$. Then $f(\Lambda\gamma_0 \cap U)$ equals the open set $f(\Lambda\gamma_0) \cap f(U)$. We may thus pick $x \in S \cap f(\Lambda\gamma_0 \cap U)$, implying that $f^{-1}(S) \cap \Lambda\gamma_0 \cap U$ is non-empty.

It remains to verify that the embedding condition is necessary. Suppose $\gamma \in \Gamma$ is such that $(\langle \gamma \rangle \times C) *_C H \to \langle \gamma, H \rangle$ is an isomorphism. Let $\mathbf{G}_1$ (resp. $\mathbf{G}_2$) denote the product of the factors of $\mathbf{G}$ over F for which the corresponding factor Ae of A is not (resp. is) a field or a totally definite quaternion algebra. Because this product decomposition is defined over F, the projections of $\mathcal{U}(\mathcal{O})$ in $\mathbf{G}_1(F \otimes_{\mathbb{Q}} \mathbb{R})$ and $\mathbf{G}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ are discrete. Since $\mathcal{D}\mathbf{G}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact, the image of $\mathcal{U}(\mathcal{O})$ in $\mathbf{G}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ is in fact finite.

As $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$, Proposition 2.7 shows that one of the kernels N_1, N_2 of the respective projections $\pi_i : \langle \gamma, H \rangle \to \mathbf{G}_i(F \otimes_{\mathbb{Q}} \mathbb{R})$, is contained in C. Of course, N_2 cannot be contained in C, otherwise the image of $\mathcal{U}(\mathcal{O})$ in $\mathbf{G}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ would contain the infinite group $(\langle \gamma \rangle \times C/N_2) *_{C/N_2} (H/N_2)$. We deduce that $N_1 \subset C$, that is, $\langle \gamma, H \rangle$ almost embeds in $\mathbf{G}_1$. Another application of Proposition 2.7 then shows that $\langle \gamma, H \rangle$ almost embeds in some factor of $\mathbf{G}_1$ over F, hence in a factor of A which is not a field nor a totally definite quaternion algebra, as claimed. □

Example 4.2. If $A = FG$ and $\mathcal{O} = RG$ for some order R in F, then by the theorem of Berman-Higman [45, Theorem 2.3.] the only torsion central units are the trivial ones (i.e. $\mathcal{U}(R).\mathcal{Z}(G)$). Thus if we take $H \leq V(RG)$, then $C = H \cap \mathcal{Z}(G)$. In particular $G * \mathbb{Z}$ exists if and only if G embeds in a simple factor and has trivial center (e.g. G is simple).

Although Corollary 4.1 is a neat existence result, it leaves open following two questions.

Question 4.3. With notations as in Corollary 4.1:

(i) How can we construct the ping-ping partner γ concretely?
(ii) When does a subgroup H embed in a simple factor?

Question (ii) will be addressed in Section 5.3. In this section we present a method for question (i) which will reduce the problem to constructing certain family of *deforming maps* (see Definition 4.4). The main result being Theorem 4.12. In Section 5 we will propose a general method to construct such maps in case that A is a group algebra.

4.1. **Deforming finite subgroups.** The aim of this section is to introduce an (explicit) linear method which allows an infinite number of ways to replace a finite subgroup $H \leq \mathrm{GL}(V)$ by an isomorphic copy. As our final aim is applications on finite subgroups in $\mathcal{U}(\mathcal{O})$, the unit group of an order $\mathcal{O}$ in a finite dimensional semisimple F-algebra (with F some field), we consider

$$A := \mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_m).$$

with V_i some finite dimensional F-vector spaces and denote $\mathcal{G} = \mathcal{U}(A)$.

Consider a subgroup $H \leq \mathcal{G}$. We want to construct a group morphism

$$T_H : H \to \mathcal{G} : h \mapsto h + \tau_h. \tag{1}$$

This will be the case if the family of elements $\tau_h \in A$ are of the following form.

Definition 4.4. Let $H \leq \mathcal{G}$. A linear map $T_H : H \to \mathcal{G}$ is called *a basic nilpotent transformation* for H if there is a set $\{\tau_h \mid h \in H\} \subseteq A$ such that $T_H(h) = h + \tau_h$ and satisfying the following three properties:

(1) $\tau_h \tau_k = 0$,
(2) $\tau_h k = \tau_h$,
(3) $\tau_{hk} = \tau_h + h\tau_k$,

for all $h, k \in H$.

If T_H is a basic nilpotent transformation, then

$$T_H(h) = h(1 + h^{-1}\tau_h) = (1 + \tau_h)h$$

with $1 + h^{-1}\tau_h$ and $1 + \tau_h$ unipotents (as $\tau_h h^{-1}\tau_h = 0$). Thus (1) can also be viewed deforming H via a family of unipotent elements.

Furthermore, note that one could have replaced the second and third property by $k\tau_h = \tau_h$ and $\tau_{hk} = \tau_h k + \tau_k$. If one considers both definitions, then the one from Definition 4.4 would be called a *right basic nilpotent transformation* and the one of the latter type a left basic nilpotent transformation. Except mentioned otherwise, we will always mean a right one.

Example 4.5. Suppose one has some $\tau_H \in A$ satisfying $\tau_H h = \tau_H$ for all $h \in H$. Then it is easily verified that by constructing $\tau_h = (1 - h)\tau_H$ for each h one obtains a (right) basic nilpotent transformation. If H is finite, one way to construct such a τ_H is by taking $x\widetilde{H} := x(\sum_{h \in H} h)$ for some $x \in A$. The choice $(\sum_{h \in H} h)x(1 - h)$ would yield a left nilpotent transformation. These constructions might be trivial, e.g. if $x \in \mathcal{Z}(A)$. If H is F-linearly independent[4] and non-central one can find some x for which the families are non-trivial.

Remark 4.6.
- The existence of a non-trivial basic nilpotent transformation T_H yield some weak restrictions on H. Among others, if A is an F-algebra, then $H \cap F.1_A = 1$. Indeed, otherwise there is some $z = \lambda 1_A \neq 1$ such that $\lambda\tau_h = \tau_h z = \tau_h$ for all $h \in H$. Since $\lambda \neq 1$, it implies that $\tau_h = 0$ as claimed.

[4]This condition is a natural one for the applications later on. Indeed, it is well-known that if H is a finite subgroup of $V(RG)$ with R a $|G|$-adapted ring, e.g. $R = \mathbb{Z}$, then the condition is satisfied.

- Notice that if $\operatorname{char}(F) \nmid |H|$ and $h \in H \cap \mathcal{Z}(A)$, then $\tau_h = 0$. Indeed, $h\tau_h = \tau_h h = \tau_h$ and hence a repeated use of properties (2) and (3) in Definition 4.4 yields $0 = \tau_1 = \tau_{h^{o(h)}} = \tau_{h^{o(h)-1}} + h^{o(h)-1}\tau_h = \ldots = o(h)\tau_h$. Hence, $\tau_h = 0$.

The following result shows the relevance of an admissible family.

Proposition 4.7. *Let $H \leq \mathcal{G}$ be a finite subgroup whose order is coprime to* $\operatorname{char} F$ *and T_H a basic nilpotent transformation. Then the associated map T_H is a monomorphism. In particular, $o(T_H(h)) = o(h)$ for any $h \in H$.*

Proof. It is a direct verification that the properties listed in Definition 4.4 yields that T_H is a homomorphism. Next, suppose that some $h \in H$ is in the kernel of T_H, so $h + \tau_h = 1$. Multiplying by $1 - h$ on the right, one obtains $h - h^2 = 1 - h$ (since $\tau_h(1 - h) = 0$). This implies that every projection of h in the components of A satisfies the polynomial $0 = 1 - 2X + X^2 = (1 - X)^2$. Hence, its minimal polynomial is either $1 - X$ or $(1 - X)^2$. In the latter case the endomorphism has a Jordan-block decomposition with each Jordan-block of size 2. These can not have finite order coprime to the order of the field. Since the element h has finite order, we therefore obtain that the endomorphism in every projection satisfies $1 - X = 0$, implying that $h = 1$. □

Remark 4.8. Notice that Proposition 4.7 is no longer true when H is an arbitrary subgroup of $\mathcal{G}$. Indeed, consider for example $A = M_2(F)$ (where F is of characteritic 0) and the endomorphism $h = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. For the infinite cyclic group $H = \langle h \rangle$, the set $\{\tau_{h^n} = \begin{bmatrix} 0 & -n \\ 0 & 0 \end{bmatrix} \mid n \in \mathbb{Z}\}$ is readily checked to be verify the conditions in Definition 4.4. However, in this case the morphism T_H is trivial.

4.2. **Constructive tools.** Till the end of this section we assume that F is any local field of characteristic 0. Thus by assumption F has an absolute value $|\cdot| : F \to \mathbb{R}^+$ such that F is locally compact with respect to the associated metric (this implies that every closed, bounded subset of F is compact). We will consider $V = F^n$ as an F-vector space with a norm $||\cdot|| : V \to \mathbb{R}^+$ derived from the absolute value on F (for example the max norm). The local compactness of F implies local compactness of V, and so again in this vector space the closed, bounded subsets are compact. Let $S = \{v \in V \mid ||v|| = 1\}$ be the unit sphere of V. As per usual, if $\sigma : V \to W$ is a transformation to another normed F-vector space W, we define the norm of this transformation as

$$||\sigma|| := \sup\{||\sigma(v)|| \mid v \in S\}.$$

If σ is linear, we have the inequality $||\sigma(v)|| \leq ||\sigma||.||v||$.

A subset of V is called projective if it is closed under multiplication with $F^\times$, so, up to presence of the zero vector, they correspond to the sets of the projective space $\mathbb{P}(V)$. We define a distance between two projective sets X and Y to be

$$d(X, Y) := \inf\{||x - y|| \mid x \in X \cap S, y \in Y \cap S\}.$$

For two non-zero vectors v and w in V we define

$$d(v, w) := d(Fv, Fw) = \inf\{||av - bw|| \mid a, b \in F : ||av|| = ||bw|| = 1\}.$$

For $v \in V$ and X a projective subset of V we set $d(v, X) := d(Fv, X)$.

One can prove the following:

Lemma 4.9 (See [28], Lemma 1.1.)**.** *With notation as defined above we have*

- *For non-zero subspaces X and Y of V, there exist elements of norm 1, $x_0 \in X$ and $y_0 \in Y$, such that $d(X, Y) = ||x_0 - y_0||$. If $X \cap Y = \{0\}$, then $d(X, Y) > 0$.*

- *The distance d defines a metric on $\mathbb{P}(V)$, for which $\mathbb{P}(V)$ has diameter*[5] ≤ 2. *Especially, it satisfies the triangular inequality for projective sets of V: $d(X,Z) \leq d(X,Y) + d(Y,Z)$ if X, Y and Z are projective sets.*
- *For non-zero vectors $v, w \in V$ we have the inequality*

$$d(v,w) \leq 2\frac{||v-w||}{||v||}.$$

If X is a projective subset of V and $\varepsilon \in \mathbb{R}^+$, then we define the (closed, projective) ε-neighbourhood of X by $\mathcal{N}_\varepsilon(X) = \{0 \neq v \in V \mid d(v,X) < \epsilon\}$, which is again a projective subset of V.

Proposition 4.10. *Let $T = h + a\tau$ be an operator on the normed F-vector space V, where $a \in F$, $h : V \to V$ arbitrary but with $||h|| < \infty$, $\tau : V \to V$ a linear, non-zero transformation with $\tau^2 = 0$ and W a bounded environment of 1 (or $W = \{1\}$) in $GL(V)$. Let $I = \operatorname{im} w_1\tau w_2$, $K = \ker w_1\tau w_2$ for $w_i \in W$ and $X \leq V$ such that $V = X \oplus K$. Moreover take $\varepsilon, \kappa \in \mathbb{R}^+$ with $\kappa \leq \frac{d(X,K)}{2}$. Then $w_1Tw_2(\mathcal{N}_\kappa(X)) \subseteq \mathcal{N}_\varepsilon(I)$ for all sufficiently large $|a|$. Moreover, this a can be chosen independently of the elements w_1 and w_2.*

Proof. This is exactly the same proof as [28, Prop. 1.2] (for notations, please refer to this proof), with the only difference in the last few calculations:

$$d(w_1Tw_2(v), I) \leq \frac{||w_1hw_2(v)||}{||aw_1\tau w_2(x)||} \leq \frac{2s||h||\ ||w_1||\ ||w_2||}{\kappa|a|}.$$

So if $|a| \geq \frac{2s||h||\ ||w_1||\ ||w_2||}{\kappa\varepsilon}$, one may conclude that $d(T(v), I) \leq \varepsilon$, proving the proposition. Remark that now this bound only depends of $\tau, X, \kappa, \varepsilon, W$ and $||h||$. Indeed, since W is bounded, we may assume $||w_1||\ ||w_2|| \leq c$, for some constant depending on W. ☐

Lemma 4.11 (See [28], Lemma 2.1)**.** *Let $T : V \to V$ be a non-singular linear transformation and let X and Y be projective subsets of V. Then*

$$d(T(X), T(Y)) \leq 2 \cdot d(X,Y) \cdot ||T|| \cdot ||T^{-1}||.$$

We are now able to give an explicit way to construct amalgamated products of finite groups. The methods are inspired by [28, Theorem 2.3] which achieve the case $H = \mathbb{Z}$.

Theorem 4.12. *Let F be a local field, V a finite-dimensional F-vector space and $H, A \leq \mathrm{GL}_n(D)$ finite subgroups. Denote $C = A \cap H$ and suppose that $[H : C] \geq 3$ or $[A : C] \geq 3$. If $T_H(h) = h + a\tau_h$ is a basic nilpotent transformation, where $a \in F^\times$, such that*

- *$\tau_h = 0 \iff h \in C$ and*
- *$g \operatorname{im}(\tau_k) \cap \ker(\tau_h) = \{0\}$ for $g \in A \setminus C, h \in H \setminus C$ and $k \in H$,*

then we have that

$$\langle A, \operatorname{im}(T_H)\rangle \cong A *_C \operatorname{im}(T_H) \cong A *_C H,$$

for all $a \in F^*$ of sufficiently large norm.

In practice, checking all the conditions $g \operatorname{im}(\tau_k) \cap \ker(\tau_h) = \{0\}$ can be difficult but luckily many are superfluous. For example, one can prove that $\ker(\tau_h) = \ker(\tau_{h^t})$ for $(o(h), t) = 1$. Building on Example 4.5 we will propose in Section 5 a way to construct a transformation T_H as in Theorem 4.12 in the case that A, H are finite subgroups of the unit group of a group ring $\mathcal{U}(FG)$.

Remark 4.13. The sufficiently large value for $|a|$ will turn out to be exactly the same as in Proposition 4.10. There an explicit lower bound can be filtered from the proof. The proof of Theorem 4.12 will also show that if F is any subfield of $\mathbb{C}$, then the result remains valid.

[5] If F is non-archimedean, then the diameter is ≤ 1. There is also the tighter bound $d(v,w) \leq \frac{||v-w||}{||v||}$.

Proof of Theorem 4.12. For ease of notation, for each $h \in H$ we will denote

$$K_h = \ker(\tau_h) \text{ and } I_h = \operatorname{im}(\tau_h). \tag{2}$$

Notice that if $C = H$, the statement is trivially satisfied. From now on we assume that $C \lneq H$. To prevent unnecessary complication, we will say that $\mathcal{N}_\varepsilon(\{0\}) = \{0\}$ for every $\varepsilon > 0$.

Let 2κ be the minimum of the finitely many distances $d(gI_k, K_h)$ for all $g \in A \setminus C, h \in H \setminus C$ and $k \in H$. Then from the assumptions and Lemma 4.9 it follows that $\kappa > 0$ and we set

$$P = \bigcup_{x \in C} \bigcup_{k \in H} \bigcup_{g \in A \setminus C} x\mathcal{N}_\kappa(gI_k) \setminus \{0\}.$$

Now let $r = \max\{2 \cdot ||g|| \cdot ||g^{-1}|| \mid g \in A \setminus \{e\}\}$, set $\varepsilon = \frac{\kappa}{r}$ and define

$$Q = \bigcup_{x \in C} \bigcup_{k \in H} x\mathcal{N}_\varepsilon(I_k) \setminus \{0\}.$$

Remark that neither P nor Q are empty. Moreover, $P \cap Q = \emptyset$. Indeed, if the intersection is not empty, then, without loss of generality, we may assume some $hv \in \mathcal{N}_\kappa(gI_l)$ for a $h \in C, g \in A \setminus C$ and $0 \neq v \in \mathcal{N}_\varepsilon(I_k)$. However, $hI_k = \operatorname{im}(\tau_{hk} - \tau_h) \subseteq K_k$ (since $\tau_h\tau_k = 0$) which imply that $d(gI_l, hI_k) \geq 2\kappa$ and $d(hI_k, hv) \leq 2 \cdot ||h|| \cdot ||h^{-1}|| d(I_k, v) \leq r\varepsilon = \kappa$ using Lemma 4.11. As such,

$$d(hv, gI_l) \geq d(gI_l, hI_k) - d(hI_k, hv) \geq 2\kappa - \kappa = \kappa,$$

which is a contradiction.

We will now play ping-pong on these two sets P and Q, using Lemma 2.1. Notice that $CP \subseteq P$ and $CQ \subseteq Q$, by construction of the sets P and Q.

We continue with proving that $(A \setminus C)\, Q \subseteq P$, so take a $g \in A \setminus C$ and $xv \in Q$ arbitrary where $x \in C$ and $0 \neq v \in \mathcal{N}_\varepsilon(I_k)$ for some $k \in H$. So, by Lemma 4.11, we have

$$d(gxv, gxI_k) \leq 2 \cdot ||gx|| \cdot ||(gx)^{-1}|| \cdot d(v, I_k) \leq r\varepsilon = \kappa,$$

proving that $gxv \in \mathcal{N}_\kappa(gxI_k) \subseteq P$ since $gx \in A \setminus C$.

Up until now, the scalar $a \in F^\times$ did not play a role, but we will choose this now such that $(\operatorname{im} T_H \setminus C)\, P \subseteq Q$. Take $T_H(h) \in \operatorname{im} T_h \setminus C$ arbitrary, and consider $T_H(h)(x\mathcal{N}_\kappa(gI_k))$ for some $x \in C, g \in A \setminus C$ and $k \in H$, assuming $I_k \neq \{0\}$. By the first condition $x = T_H(x)$ and so we see that $T_H(h)(x\mathcal{N}_\kappa(gI_k)) = T_H(hx)(\mathcal{N}_\kappa(gI_k))$ and that $T_H(hx) \in \operatorname{im} T_h \setminus C$. As such, $\tau_{hx} \neq 0$ and $gI_k \cap K_{hx} = \{0\}$. Now, we may use Proposition 4.10 applied to the operator $T_H(hx)$ and gI_k as subset of a complement X of K_{hx} to find an $a \in F$ of large enough absolute value such that

$$T_H(h)(x\mathcal{N}_\kappa(gI_k)) = T_H(hx)(\mathcal{N}_\kappa(gI_k)) \subseteq \mathcal{N}_\varepsilon(I_{hx}) \subseteq Q.$$

Since there are only a finite amount of quadruples $(x, g, h, k) \in (\operatorname{im} T_H \setminus C) \times (A \setminus C) \times H^2$, one obtains an element $a \in F$ such that this inclusion is true for every such quadruple. This shows that $(\operatorname{im} T_H \setminus C)\, P \subseteq Q$.

Because of the extra assumption that $|A : C| \geq 3$ or $|H : C| \geq 3$, we may now use Lemma 2.1 to obtain the result. □

Remark 4.14. Note that the conditions from Theorem 4.12 imply that

$$A \cap F.1_V = H \cap F.1_V \subseteq C.$$

Indeed, if g is a scalar operator then the second condition would otherwise imply that $g \operatorname{im}(\tau_k) = \operatorname{im}(\tau_k) \subseteq \ker(\tau_k)$ which is always satisfied as $\tau_k^2 = 0$. In particular, combined with the first, $\tau_k = 0$ for all $k \in H \setminus C$ in that case. Similarly, if $h \in H$ is central, then $\tau_h = 0$ by Remark 4.6 and hence $h \in C$ by the first condition. Both cases would yield a trivial amalgamated product.

Remark 4.15. Theorem 4.12 also holds when the groups A and H are subgroups of $\mathrm{PGL}_n(D)$. Indeed, the proof uses only projective sets and as such can be adapted suitably.

There is certainly a restriction on the existence of free products $A * H$ in $\mathrm{GL}_n(D)$. For example, if $A = H$ then $H * H$ exists if and only if $H * \langle t \rangle$ exists for some $t \in \mathrm{GL}_n(D)$. Following Corollary 4.1 the latter exists exactly when H contains no scalar matrices. Consequently, when $\langle A, H^x \rangle$ is finite and not intersecting the center for some $x \in \mathrm{GL}_n(D)$, then there is some $t \in \mathrm{GL}_n(D)$ yielding a subgroup $A * H^{xt}$. In fact the results in [57, 28] can be reformulated to say that H^{xt} is obtained as $\mathrm{im}(T_H)$ for some basic nilpotent transformation.

The case where $\langle A, H^x \rangle$ is infinite for any conjugated H^x of H seems much more difficult to understand. Note that this can happen as shown for example by $GL_2(\mathbb{Z}) \cong D_8 *_{C_2 \times C_2} D_{12}$. Even more, $\langle A, H^x \rangle$ will generically be infinite. Nevertheless it is now natural to ask the following:

Question 4.16. Let D be a finite division F-algebra with F a field of characteristic 0. Let A and H be finite subgroups of $\mathrm{GL}_n(D)$. Does a copy of $A *_{A \cap H} H$ exists in $\mathrm{GL}_n(D)$? If yes and if $H \cap F.1 = \{1\}$, can it be obtained as an $\langle A, \mathrm{im}(T_H) \rangle$, as in Theorem 4.12?

5. Generic constructions of amalgams and the embedding property for group rings

In Section 4, given a basic nilpotent transformation as in Definition 4.4 we have proposed a constructive way to obtain free products of finite groups in the unit group of an order in a finite semisimple algebra A. From now on we will focus on the case that A is a group ring FG and $\Gamma = \mathcal{U}(RG)$ for R an order in F. This choice of semisimple algebra and Zariski dense subgroup has the advantage to yield, using the basis G, natural candidates of ping-pong partners. The reason being that finite subgroups of $\mathcal{U}(RG)$ are R-linearly independent by a theorem of Cohn and Levingstone. More precisely, in Section 5.1 we develop further the nilpotent transformation from Example 4.5, see Definition 5.1, which is inspired from the construction of (shifted) bicyclic units. In Conjecture 5.5 we formulate that they satisfy the necessary properties to produce an amalgam as in Theorem 4.12. In particular we address the first part of Question 4.3 for group rings.

Subsequently, and most importantly, in Section 5.2 we prove that profinitely-generically two (shifted) bicyclic units generate a free group. As a corollary of all the work done we can precisely say when a given finite subgroup has a bicyclic unit as a ping-pong partner.

Finally, in Section 5.3 we discuss the second part of Question 4.3. For instance in Theorem 5.16 we obtain that a cyclic subgroup always satisfies the embedding condition from Corollary 4.1. Consequently, we get in Corollary 5.20 that a copy of $C_{o(h)} *_C C_{o(h)}$ with $C = \langle h \rangle \cap \mathcal{Z}(G)$ always exist.

Assumption: For the remainder of this section R is a commutative Noetherian domain and F is its field of fractions.

5.1. Concrete constructions and a conjecture on amalgams.

We will now apply the construction from Section 4.1, more precisely example 4.5, to the case that $A = FG$ and finite subgroups in

$$V(RG) := \{\alpha \in \mathcal{U}(RG) \mid \epsilon(\alpha) = 1\}$$

where $\epsilon : FG \to F : \sum_i a_i g_i \mapsto \sum a_i$ is the augmentation of the group algebra. Note that $\mathcal{U}(RG) = \mathcal{U}(R) \,.\, V(RG)$. The advantage of $V(RG)$ is that its finite subgroups are R-linear independent by a result of Cohn-Livingstone[6] [15].

[6] In [15] the result is only shown for number fields and their ring of integers. However the proof of [17, Corollary 2.4] combined with the general version of Berman's theorem in [64, Theorem III.1], yield the necessary fact.

Definition 5.1. Let G be a finite group, $H \leq \mathcal{U}(RG)$ a finite subgroup and $x \in RG$. Then the maps

$$b_{x,H} : H \to \mathcal{U}(RG) : h \mapsto h + (1-h)x\widetilde{H}$$

and

$$b_{H,x} : H \to \mathcal{U}(RG) : h \mapsto h + \widetilde{H}x(1-h)$$

where $\widetilde{H} := \sum_{h \in H} h$ are called the *Bovdi maps* associated to H and x. An element of $\mathcal{U}(RG)$ of the form $b_{x,H}(h)$ or $b_{H,x}(h)$ will be called a *shifted bicyclic unit.*

In case that $H = \langle h \rangle$ is cyclic and $x \in G$ such units $b_{x,\langle h \rangle}(h)$ and $b_{\langle h \rangle,x}(h)$ have been called Bovdi units in [37], in honor of Victor Bovdi who proposed such elements in that case. In [52] the elements have been rebaptised to shifted bicyclic units. Recall that *bicyclic units* are elements of the form

$$b_{\tilde{h},g} = 1 + (1-h)g\widetilde{h} \text{ and } b_{g,\tilde{h}} = 1 + \widetilde{h}g(1-h)$$

for $g, h \in G$. Note that one can rewrite $b_{x,H}(h) = h(1 + (1-h)h^{-1}x\widetilde{H})$. In particular $b_{g,\langle h \rangle}(h) = hb_{h^{-1}g,\tilde{h}}$ are slight (torsion) adaptations of bicyclic units. As the name in [52] indeed reflects their nature, we will also use that terminology.

Some of the important basic properties of the shifted bicyclic units are the following.

Proposition 5.2. *Let G be a finite group, $H \leq V(RG)$ a finite subgroup and $x \in RG$. Then the following holds:*

1. *The Bovdi maps are monomorphisms for any choice of H and x.*
2. *The groups H and $\mathrm{im}(b_{x,H})$ (and $\mathrm{im}(b_{H,x})$) are F-conjugate, i.e. there exists an $\alpha \in \mathcal{U}(FG)$ such that $\alpha^{-1}\,\mathrm{im}(b_{x,H})\alpha = H$.*
3. *If $H \leq G$ and $x \in G$, then $\mathrm{Im}(b_{x^{-1},H}) \cap \mathrm{Im}(b_{H,x}) = H \cap H^x$.*

Proof. From Proposition 4.7 and Example 4.5 we know that the Bovdi maps are monomorphisms for any choice of H and x.

Concerning the second statement, two isomorphic finite subgroups $H_1 \cong_\varphi H_2$ of $\mathcal{U}(RG)$ are F-conjugate if $\chi(h_1) = \chi(\varphi(h_1))$ for each irreducible complex character χ of G, see([65, Lemma 37.5 and Lemma 37.6], or [18, Lemma 2.6]. Now if we take $h \in H$. Because $(1-h)x\widetilde{H}$ is nilpotent, it is clear that $\chi(h) = \chi(h) + \chi((1-h)x\widetilde{H}) = \chi(b_{x,H}(h))$, confirming the claim.

For statement (3) note that $H \cap H^g = \{h \in H \mid [h, g^{-1}] \in H\}$. Therefore if $h \in H \cap H^g$, then $B_{g^{-1},H}(h) = h + g^{-1}(1-h)[h,g^{-1}]\widetilde{H} = h$. Similarly $h = B_{H,g}(h)$ and so $h \in \mathrm{Im}(b_{x^{-1},H}) \cap \mathrm{Im}(b_{H,x})$. Conversely, suppose that

$$h + (1-h)g^{-1}\widetilde{H} = k + \widetilde{H}g(1-k)$$

for some $h, k \in H$. In other words,

$$h - k + g^{-1}\widetilde{H} - hg^{-1}\widetilde{H} - \widetilde{H}g + \widetilde{H}gk = 0. \tag{3}$$

If $g \in H$, then the converse inclusion trivially holds, so suppose $g \notin H$. By Cohn-Livingstone's result finite subgroups of $\mathcal{U}(RG)$ are $R-$linear independent, thus we will look at the support of the elements. Note that $h \notin \mathrm{Supp}\{hg^{-1}\widetilde{H}\} \cup \mathrm{Supp}\{\widetilde{H}g\} \cup \mathrm{Supp}\{g^{-1}\widetilde{H}\} \cup \mathrm{Supp}\{\widetilde{H}gk\}$ as otherwise $g \in H$. Thus $h = k$. We will proof that $h \in H \cap H^g$. For this take $g^{-1}l \in g^{-1}\widetilde{H}$ which by (3) must cancel with either an element of the form $hg^{-1}t$ or tg for $t \in H$. In the former case $h = (lt^{-1})^g$, as desired. Thus we may suppose that $\mathrm{Supp}\{g^{-1}\widetilde{H}\} = \mathrm{Supp}\{\widetilde{H}g\}$. In particular $g \in g^{-1}H$, i.e. $g^2 \in H$. On this turn this entails that $g^{-1}h \in Hg$, hence also $gh = g^2g^{-1}h \in Hg$. This finishes the proof. □

The Bovdi maps can be used to construct generically several types of subgroups of $\mathcal{U}(RG)$. For example, using other terminology, in [37, Prop. 3.2.] they were used to produce solvable subgroups and free subsemigroups. Another construction is the one below. Recall that by $I(RG)$ we denote the *kernel of the augmentation map* ϵ as a ring morphism. Moreover,

$$I(RG) = \sum_{g\in G}(1-g)RG = \sum_{g\in G} R(1-g).$$

Proposition 5.3. *Let G be a finite group, $H \leq G$, $g \in G$ and set $C = H \cap H^g$. Then*

$$\langle H, \mathrm{im}(b_{g,H})\rangle \simeq I(R[H/C]) \rtimes H,$$

where H acts on $I(R[H/C])$ by left multiplication by inverses. In particular it is abelian-by-finite.

When C is not normal in H, the group $I(R[H/C])$ is meant to mean the kernel of the R-module morphism ϵ, which element wise is the same as the ring morphism ϵ, between the R-modules $R[H/C]$ and R.

Proof. For notation's sake, put $b = b_{g,H}$. Set $U = \langle H, b(H)\rangle \leq \mathcal{U}(RG)$. Remember that a shifted bicyclic unit is the product of a (generalized) bicyclic unit and an element of H:

$$b(h) = h + (1-h)g\widetilde{H} = (1 + (1-h)g\widetilde{H})h = b_h h;$$

where $b_h := 1 + (1-h)g\widetilde{H}$. So,

$$U = \langle h, b_k \mid h, k \in H\rangle.$$

Define $N = \langle b_k \mid k \in H\rangle$. We will first show that N is a normal complement of H in U and thus $U \simeq N \rtimes H$. Recall that $b_h^n = 1 + n(1-h)g\widetilde{H}$ and hence b_h is a torsion unit if and only if it is equal to 1 which happens exactly when $h^g \in H$. In particular N and H have trivial intersection. Also from the previous follows that N consists exactly of the elements of the form $b_a := 1 + ag\widetilde{H}$ with $a \in I(RH)$. Using this remark we see that N is normal:

$$b_a^x = x^{-1}(1 + ag\widetilde{H})x = 1 + x^{-1}ag\widetilde{H} = b_{x^{-1}a} \in N. \tag{4}$$

for all $x \in H$ and $a \in I(RH)$.

It remains to prove that N is isomorphic to $I(R[H/C])$. Clearly $b_{a_1}b_{a_2} = b_{a_1+a_2}$ for all $a_1, a_2 \in I(RH)$ so that we have a group epimorphism $\varphi\colon I(RH) \to N\colon a \mapsto b_a = 1 + ag\widetilde{H}$. Note that for $x, y \in H$ we have $\mathrm{Supp}(xg\widetilde{H}) \cap \mathrm{Supp}(yg\widetilde{H}) \neq \emptyset$ if and only if $xg\widetilde{H} = yg\widetilde{H}$ if and only if $xC = yC$. Thus

$$\varphi\left(\sum_{x\in H} a_x x\right) = 1 + \sum_{hC\in H/C}\left(\sum_{x\in hC} a_x\right) hg\widetilde{H},$$

and hence

$$\mathrm{Ker}(\varphi) = \bigoplus_{t\in T} tI(RC),$$

for some T a left-transversal of C in H and $N \simeq I(R[H/C])$.

Finally note that if we identify N with $I(R[H/C])$ then H acts on $I(R[H/C])$ via $\varphi\colon H \to \mathrm{Aut}(I(R[H/C]))\colon h \mapsto (a \mapsto h^{-1}a)$ by (4). □

The proof of Proposition 5.3 shows that the group $\langle 1 + (h-1)g\widetilde{H} : h \in H\rangle$ is a free-abelian group of rank $|H : H \cap H^g| - 1$. In particular, if $H \cap H^g = 1$, $\langle H, B_{-g,H}(H)\rangle \simeq I(RH) \rtimes H$ yields a free-abelian subgroup of rank $|H| - 1$.

Corollary 5.4. *Let G be a finite group and H a cyclic subgroup of G of prime order. If $g \in G$ does not normalise H then $\mathcal{U}(RG)$ contains a subgroup isomorphic to $\mathbb{Z}^{p-1} \rtimes C_p$. In particular, if $p = 2$, then $\mathcal{U}(RG)$ contains*

$$\langle C_2, B_{g,C_2}(C_2)\rangle \cong \mathbb{Z} \rtimes C_2 \cong C_2 * C_2,$$

the infinite dihedral group.

Remark. In general the existence of an abelian subgroup $H \leq G$ yields a free-abelian subgroup $F \leq \mathcal{U}(\mathbb{Z}H) \leq \mathcal{U}(\mathbb{Z}G)$ of rank $e = \frac{1}{2}(|H| + 1 + n_2 - 2\ell)$, where n_2 is the number of involutions in H and ℓ the number of cyclic subgroups of H, cf. [59, Exercise 8.3.1] or [40, Theorem 7.1.6.]. Corollary 5.4 therefore yields a larger than usually expected free-abelian subgroup.

The part of Corollary 5.4 for the prime 2 also suggests that it might be possible to make free products of finite groups using appropriate Bovdi maps. This is further supported with reformulating some results in the literature in terms of admissible families as in Theorem 4.12, see Example 4.5. All this gives evidence for the following which is a precise version of Question 4.16 in case of FG.

Conjecture 5.5. *Let $H \leq G$ be finite groups such that H has an almost embedding in a simple factor of $\mathcal{U}(FG)$. Further let $g \in G$ and denote $C = H \cap H^g$, then*

$$\langle \mathrm{im}(b_{g,H}), \mathrm{im}(b_{H,g^{-1}}) \rangle \cong H *_C H \cong \langle \mathrm{im}(b_{g,H}), \mathrm{im}(b_{g,H})^* \rangle$$

where $(\cdot)^*$ is the canonical involution on FG.

If G is nilpotent of class 2, $H \cong C_n$ and $g \in G$ such that $H \cap H^g = 1$, then [37, Theorem 4.1] shows that the conditions of Theorem 4.12 are satisfied and so $H * H$ can be constructed in the conjectured way via Bovdi maps. If n is prime, this was also obtained for arbitrary (finite) nilpotent groups. In all these cases an explicit embedding of H in a simple component of $\mathbb{Q}G$ was constructed. Recently Marciniak - Sehgal [52] were able to drop the condition on n without the use of such an embedding. The literature on constructing copies of F_2 using bicyclic units is much richer as will be recalled in the next section.

Remark. Note that the condition that H must have an embedding in a simple component is necessary by Proposition 2.7. Also, the reason why the amalgamated subgroup needs to contain C is the third part of Proposition 5.2. Note that this issue exactly corresponds to the first extra condition for a basic nilpotent transformation in Theorem 4.12.

Remark 5.6. One might hope to generalize Proposition 5.3 to a result where H is replaced by a conjugate. However, known instance of Conjecture 5.5 combined with the second part of Proposition 5.2 seem to say that such generality doesn't hold.

5.2. **Bicyclic units generically play ping-pong.** Consider in RG all elements of the form

$$b_{\tilde{h},x} = 1 + (1-h)x\widetilde{h} \text{ and } b_{x,\tilde{h}} = 1 + \widetilde{h}x(1-h) \tag{5}$$

with $x \in RG$ and $\widetilde{h} := \sum_{i=1}^{o(h)} h^i$. As $(1-h)\widetilde{h} = 0 = \widetilde{h}(1-h)$, all elements in (5) are unipotent units. The elements in the group

$$\mathrm{Bic}(G) := \langle b_{\tilde{h},x}, b_{x,\tilde{h}} \mid x \in RG \rangle$$

are called *bicyclic units.*

For many years an overarching belief in the field of group rings has been that two bicyclic units should generically generate a free group:

Conjecture 5.7. *Let G be a finite group and $\alpha \in \mathrm{Bic}(G)$. Then the set $\{\beta \in \mathrm{Bic}(G) \mid \langle \alpha, \beta \rangle \cong \langle \alpha \rangle * \langle \beta \rangle\}$ is 'large' in* $\mathrm{Bic}(G)$.

The above conjecture has been intensively investigated for $\mathbb{Z}G$. See [31] for a quit complete survey till 2013 and also [28, 30, 29, 32, 43, 63] and the references therein. The main application of Theorem 3.23 yields a concrete version of Conjecture 5.7, modulo a deformation to a shifted bicyclic unit. We also obtain a variant for a given image of a Bovdi-map. For this one needs to consider the following set

$$\mathrm{PCI}_{fp}(FG) = \{e \in \mathrm{PCI}(FG) \mid Ge \text{ is not fixed point free}\}.$$

The condition that Ge is not fixed point free boils down to say that there exists an element $g \in G$ such that $\widetilde{g}$ is non-zero.

Theorem 5.8. *Let F be a number field and R its ring of integers. Further let $H \leq G$ be finite groups and $\alpha = 1 + (1-h)x\widetilde{H}$ a bicyclic unit for some $h \in H$ and $x \in RG$. Then*

$$\mathcal{P}(\alpha) := \{\beta \in \mathrm{Bic}(G) \mid \langle \alpha h, \beta \rangle \cong \langle \alpha h \rangle * \langle \beta \rangle\}$$

is a profinitely dense subset in $\mathrm{Bic}(G)$*. Moreover if $H \cap \ker(\pi_e) \leq \mathcal{Z}(G)$ for some $e \notin \mathrm{PCI}_{fp}(FG)$, then the same holds for $\mathrm{Im}(b_{x,H})$ instead of α.*

A profinitely dense subset is also Zariski-dense [55, Proposition 2.3], hence theorem 5.8 gives a concrete interpretation of 'large' in Conjecture 5.7 for two of the natural topologies.

Remark 5.9. The condition that $e \notin \mathrm{PCI}_{fp}(FG)$ can be weakened by enlarging $\mathrm{Bic}(G)$. More precisely, consider $\mathcal{U}(RG)_{un} = \{\alpha \in \mathcal{U}(\mathbb{Z}G) \mid \alpha \text{ is unipotent }\}$. The proof of Theorem 5.8 will yield that if $H \cap \ker(\pi_e) \leq \mathcal{Z}(G)$ for some primitive central idempotent e such that FGe is not a division algebra, then $\{\beta \in \langle \mathcal{U}(RG)_{un} \rangle \mid \langle \alpha, \beta \rangle \cong \langle \alpha \rangle * \langle \beta \rangle\}$ is profinitely dense in $\langle \mathcal{U}(RG)_{un} \rangle$.

We first need the following crucial lemma relating Conjecture 5.7 to the conjecture of de la Harpe and in particular allowing to use Theorem 3.23.

Lemma 5.10. *For any finite group G the group* $\mathrm{Bic}(G)$ *is Zariski-dense in* $\mathrm{SL}_1(RG)f$ *with $f = \sum_{e \in \mathrm{PCI}_{fp}(FG)} e$.*

Now we can proceed to the proof of the main theorem.

Proof of Theorem 5.8. By the Theorem of Berman-Higman all torsion central units are trivial. Hence, the only central matrices in He are contained in B. Using **??** we obtain that $\langle He, e+\tau' \rangle \cong H *_{Be} C_\infty.Be$ for some $\tau' \in \mathbb{Q}Ge$ with $\tau'^2 = 0$. Since $\mathbb{Z}G$ is an order in $\mathbb{Q}G$ there exists a positive integer v such that $\tau = v\tau' \in \mathbb{Z}G \cap \mathbb{Q}Ge$. Clearly $\langle H, 1+\tau \rangle e = \langle He, e+\tau \rangle \cong H *_{Be} C_\infty.Be$. It follows that $\langle H, 1+\tau \rangle \cong H *_B C_\infty.B$. Next, due to the assumptions it follows from [40, Corollary 11.2.1 and Theorem 11.2.5] that the bicyclic units of $\mathcal{U}(\mathbb{Z}G)$ contain a subgroup of finite index in $1-e+\mathrm{SL}_1(\mathbb{Q}Ge)$. Since $1+\tau$ is in this group, replacing if necessary $1+\tau$ by $(1+\tau)^w = 1+w\tau$ we obtain the desired form of the ping-pong partner. ☐

Corollary 5.11. *Let $H \leq V(\mathbb{Z}G)$. If there exists $e \in PCI_{nc}(G)$ such that $\mathbb{Q}Ge$ is not exceptional, Ge is not fixed point free and $\ker(\pi_e) \cap H = 1$. Then there exists a unit $b \in \mathrm{Bic}(G)$ such that $\langle H, b \rangle \cong H *_B \langle b \rangle.B$ with $B = H \cap \mathcal{Z}(G)$.*

Proof. By the Theorem of Berman-Higman all torsion central units are trivial. Hence, the only central matrices in He are contained in B. Using **??** we obtain that $\langle He, e+\tau' \rangle \cong H *_{Be} C_\infty.Be$ for some $\tau' \in \mathbb{Q}Ge$ with $\tau'^2 = 0$. Since $\mathbb{Z}G$ is an order in $\mathbb{Q}G$ there exists a positive integer v such that $\tau = v\tau' \in \mathbb{Z}G \cap \mathbb{Q}Ge$. Clearly $\langle H, 1+\tau \rangle e = \langle He, e+\tau \rangle \cong H *_{Be} C_\infty.Be$. It follows that $\langle H, 1+\tau \rangle \cong H *_B C_\infty.B$. Next, due to the assumptions it follows from [40, Corollary 11.2.1 and Theorem 11.2.5] that the bicyclic units of $\mathcal{U}(\mathbb{Z}G)$ contain a subgroup of finite index in $1-e+\mathrm{SL}_1(\mathbb{Q}Ge)$. Since $1+\tau$ is in this group, replacing if necessary $1+\tau$ by $(1+\tau)^w = 1+w\tau$ we obtain the desired form of the ping-pong partner. ☐

In particular we obtain the following result for simple groups.

Corollary 5.12. *Let G be a non-abelian finite simple group and $1 \neq H \leq G$. Then there exists a bicyclic unit $b \in \mathrm{Bic}(G)$ such that*

$$\langle H, b\rangle \cong H * \langle b\rangle.$$

Proof. In case G is simple, the morphism $G \to Ge_i$ is clearly an embedding for every primitive central idempotent such that $\mathbb{Q}Ge_i \cong M_{n_i}(D_i)$ with $n_i \geq 2$. Furthermore since G is assumed non-abelian simple we claim that there exists an idempotent $e \in \mathrm{PCI}_{nc}(G)$ such that $\mathbb{Q}Ge$ is non-exceptional. Suppose the contrary, then on one hand for all $e \in \mathrm{PCI}_{\neq 1}(G)$ the $\dim \mathbb{Q}Ge$ would be a 2-power and on the other hand there is only one 1×1-component (since by Amitsur classification the multiplicative group of a division algebra does not contain a non-abelian simple group and because $G^{ab} = 1$) which moreover correspond to the trivial representation and hence $\mathbb{Q}$. So all together in this case $|G| = \dim_{\mathbb{Q}} \mathbb{Q}G$ would be odd, hence solvable by Feit-Thompson[7] and thus G would have to be isomorphic to C_p, a contradiction. The conclusion now follows from Corollary 5.11. □

5.3. **On the embedding condition for group rings.** In this section we consider the group algebra $\mathbb{Q}G$ and wish to understand when a finite subgroup H of $\mathcal{U}(\mathbb{Z}G)$ has the embedding condition from Corollary 4.1 to find a ping-pong partner for H.

5.3.1. *Faithful irreducible embedding over different fields.* The existence of irreducible faithful complex representations for finite groups has already been intensively studied, see [72, Section 2] for a survey. We however need to understand the existence of such representations for smaller fields. So let F be a field with $\mathrm{char}(F) = 0$ and let $FG \cong \prod_{i=1}^{q} \mathrm{M}_{n_i}(D_i)$ be its Wedderburn-Artin decomposition. For every $e \in \mathrm{PCI}(FG)$ we will denote $FGe \cong \mathrm{M}_{n_e}(D_e)$ the associated simple quotient and by

$$\pi_e : \mathcal{U}(FG) \twoheadrightarrow FGe \cong \mathrm{GL}_{n_e}(D_e)$$

the map induced by the projection onto FGe.

Definition 5.13. A finite subgroup $H \in U(FG)$ is said to have *a f.i.r with respect to G and F* if there exists a primitive central idempotent e of FG such that $H \cap \ker(\pi_e) = 1$. If $H = G$, than we say that G has *f.i.r. over F*.

We will use the following notation for the set of primitive central idempotents yielding a f.i.r. for H:

$$\mathrm{Emb}_{G,F}(H) = \{e \in \mathrm{PCI}(FG) \mid H \cap \ker(\pi_e) = 1\}. \tag{6}$$

If F is clear from the context, then we will simply write $\mathrm{Emb}_G(H)$. Using well-known results over $\mathbb{C}$ one readily obtains the following.

Lemma 5.14. *Let G be a finite group, $F \subseteq L$ be fields of characteristic 0 and $H \leq \mathcal{U}(FG) \subset \mathcal{U}(LG)$ a finite subgroup. Then*

(1) If G a f.i.r. over F, then $\mathcal{Z}(G)$ is cyclic.
(2) If H has f.i.r. with respect to G and L, than also to G and F.
(3) If G is nilpotent, then it has a f.i.r. over F if and only if $\mathcal{Z}(G)$ is cyclic.

Moreover

Proof. Since finite subgroups of a field are cyclic and $\mathcal{Z}(G)e \subseteq \mathcal{Z}(FGe)$ for any $e \in \mathrm{PCI}(FG)$ (as Ge generates the simple component as F-vector space), it follows that $\mathcal{Z}(G)$ is cyclic. Next, note that $\mathbb{C}G \cong \mathbb{C} \otimes_F FG \cong \oplus_{f \in \mathrm{PCI}(FG)} (\mathbb{C} \otimes_F FGf)$ and $\mathbb{C} \otimes FGf$ might be only semisimple over $\mathbb{C}$. Clearly the kernel of the projection to any $\mathbb{C}$-simple component of $\mathbb{C}Gf$ contains $\ker(\pi_f)$. Therefore if there exists an $e \in \mathrm{PCI}(LG)$ such that $H \cap \ker(\pi_e) = 1$, then $H \cap \ker(\pi_f) = 1$ for some $f \in \mathrm{PCI}(FG)$. In other words, the second

[7] The use of the odd-order theorem can be avoided by instead looking more in depth into [4, Appendix A]. By doing so one notices the absence of simple groups in this list.

assertion holds. For the last part it now suffices to recall that if $\mathcal{Z}(G)$ is cylic and G nilpotent then it has a f.i.r. over $\mathbb{C}$. □

Another handy sufficient condition to have a f.i.r. over $\mathbb{C}$ is that all Sylow subgroups have a cyclic center [36, Exercise 5.25].

Lemma 5.14 gives the existence of an embedding in a simple factor, however it doesn't indicate how to find the representation, or alternatively the necessary primitive central idempotent.

Example 5.15. Let G be a finite nilpotent group with cyclic center. Then by Lemma 5.14 it has a f.i.r. over $\mathbb{Q}$. This can be constructed as following: write $\mathcal{Z}(G) = \langle z_1 \rangle \times \ldots \times \langle z_n \rangle$ with each z_i of order $p_i^{n_i}$ where $p_1, \ldots, p_n$ are distinct prime numbers and n_i is a positive integer. Let $c_i = z_i^{p_i^{n_i - 1}}$, an element of order p_i. Then $f = \prod_{i=1}^{n} (1 - \widehat{c_i})$ is a central idempotent of $\mathbb{Q}G$ and thus $f = \sum_{i=1}^{t} e_i$ is a sum of primitive central idempotents of $\mathbb{Q}G$.

Claim: the natural epimorphism $G \to Ge_i \leq \mathcal{U}((\mathbb{Z}G)e_i)$ is an embedding[8] for each i. Moreover, if there is some $H \leq G$ with $H \cap \mathcal{Z}(G) = 1$, then Ge_i is not a fixed point free group. In particular, $\mathbb{Q}Ge_i$ is not a rational division algebra.

Proof. To see the first part suppose the contrary. Then let $G_{e_i} = \{y \in G \mid ye_i = e_i\}$ be a non-trivial normal subgroup of G for some i. Hence $G_{e_i} \cap \mathcal{Z}(G) \neq \{1\}$ and thus G_{e_i} contains a c_j for some $1 \leq j \leq n$. Hence, $e_i = fe_i = \left(\prod_{k=1, k \neq j}^{n} (1 - \widehat{c_k}) \right) (1 - \widehat{c_j}) e_i = 0$, a contradiction.

For the second part, suppose G is non-abelian. The last part will follow from the second as finite subgroups of D^* are fixed point free (e.g. see [40, pg 347]). Recall that fixed point free groups are exactly the Frobenius complements, [40, Prop. 11.4.6.]. As G is nilpotent, if this would be the case, then by [40, Corollary 11.4.7.] $G \cong Ge_i$ would be either cyclic or isomorphic to $Q_{2^t} \times C_p$ for some prime $p \neq 2$.

Consider the second case. Recall that $\mathcal{Z}(Q_{2^t}) \cong C_2$, whose generator we denote by -1. Then, if H contains some $(x, c) \in Q_{2^t} \times C_p$ with $x \neq 1$, we can take a 2-power q such that $(x, c)^q = (-1, c^q) \in H \cap \mathcal{Z}(G)$, a contradiction. Thus $H \leq C_q \subset \mathcal{Z}(G)$, also a contradiction. □

Note that, in view of the proof, we could also have supposed that G is a non-abelian p-group. In fact if $G \cong Q_{2^t} \times C_p$, then it might happen that $\mathbb{Q}Ge_i$ is a division algebra. Nevertheless in this case $G/\langle -1 \rangle \cong D_{2^{t-1}} \times C_p$ which embeds in a simple factor over $\mathbb{Q}$.

5.3.2. *Embeddings for cyclic subgroups and its corollaries.* We will now proof that a cyclic subgroup of G always embeds in a suitable simple component of $\mathbb{Q}G$. More generally we proof this for any $h \in \mathcal{U}(\mathbb{Z}G)$ that is conjugated inside $\mathbb{Q}G$ to an element of G, see Remark 5.17 for which large classes of groups this always holds.

Theorem 5.16. *Let G be a finite group and $h \in \mathcal{U}(\mathbb{Z}G)$ torsion. Suppose that one of the following cases hold:*

(I) $h^\alpha \in \pm G$ for some $\alpha \in \mathbb{Q}G$.

(II) $o(h)$ is a prime power.

If $\langle h \rangle \cap \mathcal{Z}(G) = 1$, then there exists some $e \in PCI(\mathbb{Q}G)$ such that $\langle h \rangle \cap \ker(\pi_e) = 1$ and $\mathbb{Q}Ge$ is neither a field nor a totally definite quaternion algebra.

Remark 5.17. Condition (I) in Theorem 5.16 is reminiscent of the first Zassenhaus conjecture. The latter states that, for finite G, any $h \in \mathcal{U}(\mathbb{Z}G)$ is conjugated in $\mathbb{Q}G$ to an

[8]Note that Ge_i is indeed a group since e_i is central

element of $\pm G$. This conjecture was recently disproved in [21]. However for large classes of groups it holds, such as *nilpotent groups* [76, 77] and *cyclic-by-abelian groups* [14]. See [53] for a survey. Thus for these classes of groups Theorem 5.16 yields that $\mathrm{Emb}_G(\langle h\rangle) \neq \emptyset$ for any $h \in \mathcal{U}(\mathbb{Z}G)$. Also, condition (II) conjecturally implies condition (I), see conjecture (p-ZC3) in [14, Section 6].

We need the following general lemma to prove Theorem 5.16.

Lemma 5.18. *Let G be a finite group and $H \leq V(\mathbb{Z}G)$ torsion such that $H \cap \mathcal{Z}(G) = 1$. If $\mathrm{Emb}_G(H) \neq \emptyset$, then there exists $e \in \mathrm{Emb}_G(H)$ such that $\mathbb{Q}Ge$ is neither a field or a totally definite quaternion algebra.*

Proof. Suppose that $\mathbb{Q}Ge$ is a field for every $e \in \mathrm{Emb}_G(H)$. In particular H can be viewed as a subgroup of a field and hence is cyclic, say $H = \langle h\rangle$. Take $p \mid o(h)$ prime and consider the element $h^{o(h)/p}$ of prime order which is in $\ker(\pi_e)$ for each $e \in \mathrm{PCI}(\mathbb{Q}G) \setminus \mathrm{Emb}_G(H)$. Thus, due to the current assumption on $\mathrm{Emb}_G(H)$, $h^{o(h)/p} = (g, 1) \in \mathbb{Q}[G/G'] \oplus \mathbb{Q}G(1-\widehat{G'})$ for some g. In particular, as $\mathbb{Q}[G/G']$ is commutative, $h^{o(h)/p}$ is central which contradicts $\langle h\rangle \cap \mathcal{Z}(G) = 1$. Thus by contradiction we may assume that there exists some $e \in \mathrm{Emb}_G(h)$ for which $\mathbb{Q}Ge$ is a totally definite quaternion algebra, say $\left(\frac{a,b}{K}\right)$.

For the sequel of the proof we fix a non-trivial element $h \in H$ and $e \in \mathrm{Emb}_G(h)$ such that $\mathbb{Q}Ge \cong \left(\frac{a,b}{K}\right)$.

Thus Ge embeds in the unit group of a finite dimensional division algebra and hence it is a Frobenius complement [68, 2.1.2, page 4]. In our case:

Claim 1: Let $\mathcal{O}$ an order in $\left(\frac{a,b}{K}\right)$. If $G \leq \mathcal{U}(\mathcal{O})$ is a finite subgroup such that $\mathrm{span}_{\mathbb{Q}}\{ge \mid g \in G\} \cong D$, then Ge is isomorphic to one of the following:

- $Q_{4m} = \langle a, b \mid a^{2m} = 1, b^2 = a^m, ba = a^{-1}b\rangle$ generalized quaternion group
- $\mathrm{SL}_2(\mathbb{F}_3), \mathrm{SU}_2(\mathbb{F}_3), \mathrm{SL}_2(\mathbb{F}_5)$

The statement of Claim 1 follows from [75, Prop. 32.4.1, Lemma 32.6.1 & Prop. 32.7.1].

It is well-known that all the groups in Claim 1 have the property that all elements of order 2 are central. Consequently if $x \in D^*$ with D a field or $\left(\frac{a,b}{K}\right)$ and $o(x) = 2$, then $x \in \mathcal{Z}(D)$ (e.g. see [75, 32.5.6 pg 599]). Therefore, recalling that by Berman-Higman $H \cap \mathcal{Z}(G) = H \cap \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$, the condition on H entails that $o(h)$ must be odd (as otherwise $1 \neq h^{o(h)/2} \in \mathcal{Z}(G)$ by the preceding).

In summary, we have obtained that if $|H|$ is even with $H \cap \mathcal{Z}(G) = 1$, then $\mathrm{Emb}_G(H)$ do not only contain a primitive central idempotent e such that $\mathbb{Q}Ge \cong \left(\frac{a,b}{K}\right)$ or a field. In case that $|H|$ is odd, the desired statement follows directely from the following:

Claim 2: All groups from Claim 1 which are not a 2-group have an irreducible representation over $\mathbb{Q}$ into a simple algebra $\mathrm{M}_2(D)$ whose kernel doesn't intersect H.

We may assume that $H \leq G$. This because there exists some $\alpha \in QG$ such that $H^\alpha e \leq Ge$ for Ge as in Claim 1. Indeed the third Zassenhaus conjecture was proven for Q_{4m} in [14], for $\mathrm{SL}_2(\mathbb{F}_5)$ in [20, Theorem 4.3] and in [19, Theorem 4.7] for $\mathrm{SU}_2(\mathbb{F}_3)$. For $\mathrm{SL}_2(\mathbb{F}_3)$ note that H, and so also He, being of odd order implies that He is cyclic, allowing to use the known first Zassenhaus conjecture for $\mathrm{SL}_2(\mathbb{F}_3)$ in [34].

Now for Q_{4m} we use the description in [40, Example 3.5.7.] of the Strong Shoda pairs and associated simple components. More precisely, consider the SSP $(G, \langle a\rangle, \langle a^d\rangle)$ with $2 \neq d \mid n$ and the associated primitive central idempotent $e(G, \langle a\rangle, \langle a^d\rangle)$. Then $\mathbb{Q}Ge(G, \langle a\rangle, \langle a^d\rangle) \cong \mathrm{M}_2(\mathbb{Q}(\zeta_d + \zeta_d^{-1}))$ with ζ_d a d-th primitive root of unity and $\langle a^d\rangle = \ker(\pi_e) \lhd G$. In particular $H \cap \ker(\pi_e) = 1$, as desired.

For the other groups [4, Table in Appendix] learns that $\mathrm{M}_2(\left(\frac{-1,-3}{\mathbb{Q}}\right))$ is a faithful irreducible component over $\mathbb{Q}$ of $\mathrm{SU}_2(\mathbb{F}_3)$ and $\mathrm{SL}_2(\mathbb{F}_5)$ and $\mathrm{M}_2(\mathbb{Q}(\sqrt{-3}))$ for $\mathrm{SL}_2(\mathbb{F}_3)$. Alternatively, it is well-known that they have a f.i.r over $\mathbb{C}$ and hence by Lemma 5.14 also over $\mathbb{Q}$. This proves Claim 2, finishing the proof. □

Next we need a lemma saying that finite cyclic subgroups have a f.i.r with respect to G and $\mathbb{C}$. We warmly thank Miquel Martínez for sharing the proof of Lemma 5.19.

Lemma 5.19. *Let G be a finite nonabelian group and $1 \neq g \in G$. Then there exists a complex irreducible character χ of G with $\chi(1) > 1$ and $g \notin \ker(\chi)$. In other words, $\langle g \rangle$ has a f.i.r with respect to G and $\mathbb{C}$.*

Proof. Assume g is in the kernel of every complex irreducible non-linear character of G. Consequently $g \notin G'$ as otherwise $g \in \bigcap_{\chi \in \mathrm{Irr}(G)} \ker(\chi) = 1$. Therefore [56, Corllary 4.10] yields that

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)} = 0. \tag{7}$$

Due to the assumption on g the latter sum can be rewritten :

$$\begin{aligned} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)} &= \sum_{\lambda \in \mathrm{Lin}(G)} \lambda(g) + |\{\chi \in \mathrm{Irr}(G) \mid \chi(1) > 1\}| \\ &= \sum_{\lambda \in \mathrm{Irr}(G/G')} \lambda(gG') + |\{\chi \in \mathrm{Irr}(G) \mid \chi(1) > 1\}| \end{aligned} \tag{8}$$

Now, for any abelian group A the following holds

$$\sum_{\lambda \in \mathrm{Irr}(A)} \lambda(g) = 0. \tag{9}$$

Indeed, first note that for $1 \neq g \in A$ there exists a $\mu \in \mathrm{Irr}(A)$ with $g \notin \ker(\chi)$ (i.e. $\mu(g) \neq 1$). Next recall that $\mathrm{Irr}(A) = \hat{A}$ is multiplicative group. With this we deduce the equation

$$\sum_{\lambda \in \mathrm{Irr}(A)} \lambda(g) = \sum_{\lambda \in \mathrm{Irr}(A)} (\mu\lambda)(g) = \mu(g) \sum_{\lambda \in \mathrm{Irr}(A)} \lambda(g)$$

which yields (9) as also $\mu(g) \neq 0$.

Finally, filling (9) and (8) in (7) we get that $|\{\chi \in \mathrm{Irr}(G) \mid \chi(1) > 1\}| = 0$, i.e that G is abelian. This is a contradication, finishing the proof. □

Proof of Theorem 5.16. By Lemma 5.18 it is enough to proof that $\mathrm{Emb}_G(\langle h \rangle) \neq \emptyset$. Therefore, by ways of contradiction, suppose that $\mathrm{Emb}_G(\langle h \rangle)$ is empty.

First suppose that $h^{\alpha} \in \pm G$ for some $\alpha \in \mathcal{U}(\mathbb{Q}G)$. As any $\ker(\pi_e)$ is an ideal $h^{\alpha} \in \ker(\pi_e)$ if and only if $h \in \ker(\pi_e)$. Furthermore, $h^{\alpha} \in \ker(\pi_e)$ excactly when $-h^{\alpha}$ is. Thus, in this case, without lose of generality we may assume that $h \in G$. Now Lemma 5.19 combined with Lemma 5.14 yields that $\mathrm{Emb}_G(\langle h \rangle) \neq \emptyset$ as desired.

Next suppose that h has prime power order. Take for every $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that $\mathbb{Q}Ge$ is not a field an element $1 \neq h^{p^l} \in \langle h \rangle \cap \ker(\pi_e)$. Thus $p^l \neq o(h)$ and l depends on e. Hence, taking the maximum of all these powers, say p^k, we know that $1 \neq \langle h^{p^k} \rangle \leq \ker(\pi_e)$ for each non-field component. Hence, considering that $h^{p^k} = 1h^{p^k} = \sum_{e \in PCI(\mathbb{Q}G)} eh^{p^k}$, and $eh^{p^k} = e$ when the component is non-commutative (by construction), we readily obtain that $h^{p^k} \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$. As such, by the Theorem of Berman-Higman, see [59, Corollary 7.1.3], $1 \neq h^{p^k} \in \mathcal{Z}(G)$, a contradiction. □

An interesting consequence of our methods is a new proof of the main existence result of $C_p * \mathbb{Z}$ by Goncalves-Passman [27] and in fact a generalisation of it to prime power order

Corollary 5.20. *Let G and $h \in \mathcal{U}(\mathbb{Z}G)$ as in theorem 5.16. Denote $C = \langle h \rangle \cap \mathcal{Z}(G)$. Then, there exists some $t \in \mathcal{U}(\mathbb{Z}G)$ such that*

$$\langle h, t \rangle \cong \langle h \rangle *_C \langle t, C \rangle \cong C_{o(h)} *_C (\mathbb{Z} \times C).$$

In particular, $C_{o(h)} *_C C_{o(h)}$ exists in $\mathcal{U}(\mathbb{Z}G)$. Moreover, $C_p * \mathbb{Z}$ exists in $\mathcal{U}(\mathbb{Z}G)$ if and only if G has a non-central element of order p.

Proof. Combining Theorem 5.16 and Corollary 4.1 we obtain an element $t \in \mathcal{U}(\mathbb{Z}G)$ of infinite order such that canonically $\langle h, t \rangle \cong \langle h \rangle *_C \langle t, C \rangle \cong C_{o(h)} *_C (\mathbb{Z} \times C)$. It is classical and easy to see that now $\langle h, h^t \rangle \cong \langle h \rangle *_C \langle h^t \rangle \cong C_{o(h)} *_C C_{o(h)}$.

Now suppose that there exists a copy of $C_p * \mathbb{Z}$ in $\mathcal{U}(\mathbb{Z}G)$. Then $\mathcal{U}(\mathbb{Z}G)$ contains a non-central element of order p. By the positive solution on the Kimmerle problem for prime order elements [46, Corollary 5.2.], this implies that G must have a non-central element of order p, yielding the sufficiency of the last part of the statement. ☐

6. Virtual structure problem for product of amalgam and HNN over finite groups

In this final section we consider the virtual structure problem which was for the first time explicitly formulated in [39] but in fact goes back to the question on 'unit theorems' by Kleinert [49].

Question 6.1 (Virtual Structure Problem)**.** Let $\mathcal{G}$ be a class of groups. Classify the finite groups G such that $\mathcal{U}(\mathbb{Z}G)$ has a subgroup of finite index lying in $\mathcal{G}$

In [39], building on [38, 42, 50, 41], Jespers-Del Rìo solved the problem for

$$\mathcal{G}_{pab} = \{\prod_i A_{i,1} * \cdots * A_{i,t_i} \mid A_{i,j} \text{ are finitely generated abelian } \}$$

where $t_i = 1$ is allowed (i.e. an abelian factor). It turns out the classification coincide with the case of products of free groups (where again $\mathbb{Z}$ is also allowed). Moreover the problem for the classes $\{A * B \mid A, B \text{ f.g. abelian } \}$ and $\{ \text{ free groups } \}$ coincide and there is only four finite groups satisfying this (in all these cases $\pm G$ has a free normal complement in $\mathcal{U}(\mathbb{Z}G)$ [38]).

We will now consider the case

$$\mathcal{G}_\infty := \{\prod_i \Gamma_i \mid \Gamma_i \text{ has infinitely many ends } \}.$$

By Stallings theorem [71, 70] a group has infinitely many ends if and only if it can be decomposed as an amalgamated product or HNN extension over a finite group. In fact we will mainly work with this characterisation. Recall that given a finitely generated group Γ, then the number of ends $e(\Gamma)$ is the defined in terms of its Cayley graph $\mathrm{Cay}(\Gamma, S)$ with S a finite generating set[9]. More precisely, $e(\Gamma)$ is the smallest number m such that for any finite set F the graph $\mathrm{Cay}(\Gamma, S) \setminus F$ has at most m infinite connected components. If no finite m exists one defines $e(\Gamma) = \infty$.

Despite that the class $\mathcal{G}_\infty$ is much larger than the aforementioned classes the virtual structure problem for it coincide.

Theorem 6.2. *Let G be a finite group. The following are equivalent:*

(1) $\mathcal{U}(\mathbb{Z}G)$ is virtually in $\mathcal{G}_\infty$,

[9]The number of ends is known to be independent of the chosen generating set.

(2) all the simple components of $\mathbb{Q}G$ *are of the form* $\mathbb{Q}(\sqrt{-d})$, *with* $d \in \mathbb{N}$, $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ *with non-zero* $a, b \in \mathbb{N}$ *or* $\mathrm{M}_2(\mathbb{Q})$ *and the latter needs to occur.*

Moreover, only the parameters $(-1,-1)$ *and* $(-1,-3)$ *can occur for* $(-a,-b)$. *Also,* $e(\mathcal{U}(\mathbb{Z}G)) = \infty$ *if and only if it virtually free if and only if* G *is isomorphic to* $D_6, D_8, Dic_3, C_4 \rtimes C_4$.

In the statement above we used the notation $D_{2n} = \langle a, b \mid a^n = 1 = b^2, a^b = a^{-1}\rangle$, $Dic_3 = \langle a, b \mid a^6, a^3 = b^2, a^b = a^{-1}\rangle$ and $C_4 \rtimes C_4 = \langle a, b \mid a^4 = b^4 = 1, a^b = a^{-1}\rangle$. That these groups are exactly the cases when $\mathcal{U}(\mathbb{Z}G)$ is virtually free is known since [38, 39], however we give a new short proof of this using amalgamated product methods and in particular Proposition 2.7.

Using the description obtained in [50, Theorem 1] in terms of simple components we indeed see that the classes correspond.

Corollary 6.3. *Let* G *be a finite group. The following are equivalent:*

(1) $\mathcal{U}(\mathbb{Z}G)$ *is virtually in* $\mathcal{G}_\infty$,
(2) $\mathcal{U}(\mathbb{Z}G)$ *is virtually a direct product of non-abelian free groups.*

Another interesting corollary of Theorem 6.2 is that if $\mathcal{U}(\mathbb{Z}G)$ is virtually in $\mathcal{G}_\infty$, then G is a cut group (i.e. $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite).

Remark 6.4. It is well known that $e(\Gamma) \in \{0, 1, 2, \infty\}$ for a finitely generated group. By definition, $e(\Gamma) = 0$ if and only if Γ is finite. Moreover, $e(\Gamma) = 2$ if and only if Γ has a subgroup of finite index isomorphic to $\mathbb{Z}$. In case $\Gamma = \mathcal{U}(\mathbb{Z}G)$ the former happens exactly when G is abelian with $\exp(G) \mid 4, 6$ or $G \cong Q_8 \times C_2^n$ for some n (see [40, Theorem 1.5.6.], as proven by Higman). The latter has not yet been recorded in the literature but fellows readily from classical methods:

Description: $e(\mathcal{U}(\mathbb{Z}G)) = 2$ if and only if $\mathcal{U}(\mathbb{Z}G)$ is $\mathbb{Z}$-by-finite if and only if G is isomorphic to C_5, C_8 or C_{12}.

Proof. The first equivalence holds for any finite generated group and is well-known [35, 24] (or [70, pg 38]). For the second, following Kleinert [47] (or [40, prop. 5.5.6]) $\mathcal{U}(\mathbb{Z}G)$ is abelian-by-finite if and only if all the simple components of $\mathbb{Q}G$ are either fields or totally definite quaternion algebras. Consequently, $\mathbb{Q}G$ has no non-trivial nilpotent elements in which case [66] tells that G is either abelian or $G \cong Q_8 \times C_2^m \times A$ with $m \geq 0$ and A an abelian group of odd order.Suppose first that $G \cong Q_8 \times C_2^m \times A$. Then $\mathbb{Q}G \cong (4m\,\mathbb{Q} \oplus m\left(\frac{-1,-1}{\mathbb{Q}}\right)) \otimes_{\mathbb{Q}} \mathbb{Q}A$. We now see that in order to obtain a copy of $\mathbb{Z}$ in $\mathcal{U}(\mathbb{Z}G)$ that his will have to come from a component of $\mathbb{Q}A$. However this component will appear at least 4 times and hence such groups are never $\mathbb{Z}$-by-finite.

Now suppose that G is abelian. By the theorem of Perlis-Walker [59, Th.3.5.4] $\mathbb{Q}G \cong \bigoplus_{d \mid |G|} a_d \mathbb{Q}(\zeta_d)$ with a_d the number of different cyclic subgroups of order d. Denote by R_d the ring of integers of $\mathbb{Q}(\zeta_d)$ and recall that by Dirichlet Unit theorem [40, Th. 5.2.4] the rank of $\mathcal{U}(R_d)$ is $\frac{\varphi(d)}{2} - 1$. A direct computation yields that $\varphi(d) \leq 4$ if and only if $d \in \{2, 3, 5, 4, 8, 10, 12\}$ with equality only for $\{5, 8, 10, 12\}$. This combined with Perlis-Walker's decomposition we see that we only have *exactly one* copy $\mathbb{Z}$ when G is C_5, C_8 or C_{12}. ☐

Consequently, it would be natural to consider the class

$$\mathcal{G}_{\neq 1} := \{\prod_i \Gamma_i \mid e(\Gamma_i) \neq 1\}.$$

With a bit more of work one can in fact prove that

$$\{G \text{ finite s.t. } \mathcal{U}(\mathbb{Z}G) \text{ is virtually-}\mathcal{G}_{\neq 1}\} = \{G \text{ finite s.t. } \mathcal{U}(\mathbb{Z}G) \text{ is virtually-}\mathcal{G}_{pab}\}. \tag{10}$$

We will now start with the proof of Theorem 6.2. This requires the following lemma that is a generalisation of [39, prop. 4.5.].

Lemma 6.5. *Let G be a finite group, D be a finite dimensional division algebra over F with* $\mathrm{char}(F) = 0$*, different*[10] *of* $\left(\frac{-2,-5}{\mathbb{Q}}\right)$*, and suppose* $\mathrm{M}_n(D)$ *with* $n \geq 2$ *is a simple component of* FG*. If* $\mathcal{O}$ *is an order in* $\mathrm{M}_n(D)$*, then* $e(\mathcal{U}(\mathcal{O})) = \infty$ *if and only if* $n = 2$ *and* $D = F = \mathbb{Q}$*.*

Proof. Suppose $e(\mathcal{U}(\mathcal{O})) = \infty$. Recall that, see [70, pg38], that if Γ_1 and Γ_2 are commensurable then $e(\Gamma_1) = e(\Gamma_2)$. Moreover, the unit group of two orders are commensurable [40, lemma 4.6.9]. Thus without lose of generality we will assume that $\mathcal{O}$ is a maximal order in $\mathrm{M}_n(D)$. It is well known that in that case $\mathcal{O} \cong \mathrm{M}_n(\mathcal{O}_{max})$ with $\mathcal{O}_{max}$ a maximal order in D. Next recall that any group with infinitely many ends has finite center (as central elements need to be in the subgroup over which the amalgam and HNN are constructed, which is now finite). Therefore, $\mathrm{SL}_n(\mathcal{O}_{max})$ has finite index in $\mathrm{GL}_n(\mathcal{O}_{max})$ and hence $\mathrm{SL}_n(\mathcal{O}_{max}))$ also has infinitely many ends. This implies that $\mathrm{SL}_n(\mathcal{O}_{max})$ has S-rank 1, with S the set of infinite places, as otherwise it has hereditarily Serre's property FA (even property T [54, 23]).

The S-rank being one means that $n = 2$ and D is either $\mathbb{Q}(\sqrt{-d})$, with $d \geq 0$ or $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ with a, b strictly positive integers (see [4, Theorem 2.10.]). Furthermore it was proven in [22] that the condition that $\mathrm{M}_2(D)$ is a component of a group algebra yields that $d \in \{0, -1, -2, -3\}$ and $(a, b) \in \{(1, 1), (1, 3), (2, 5)\}$. All these division algebras are (right norm) Euclidean and due to this have a unique maximal order (see [4, remark 3.13]), which we denote $\mathcal{O}_D$. By assumption $(a, b) = (2, 5)$ doesn't occur. Now, following [4, Theorem 5.1] $\mathrm{GL}_2(\mathcal{O}_D)$ has property FA except if $D = \mathbb{Q}$ or $\mathbb{Q}[\sqrt{-2}]$. In case of $D = \mathbb{Q}(\sqrt{-2})$ one can use the amalgam decomposition of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ given in [25, Theorem 2.1] to see that the group doesn't admit a splitting over a finite group. Finally, $\mathrm{GL}_2(\mathbb{Z}) = D_8 *_{C_2 \times C_2} D_{12}$ and hence $e(\mathrm{GL}_2(\mathbb{Z})) = \infty$, finishing the proof. □

We now proceed to the main proof.

Proof of Theorem 6.2. It is well known (e.g. see [70, pg38]) that if Γ_1 and Γ_2 are two groups such that $\Gamma_1 \cap \Gamma_2$ has finite index in the both (i.e. the Γ_i are commensurable), then $e(\Gamma_1) = e(\Gamma_2)$. Also if N is a finite normal subgroup, then $e(\Gamma_1) = e(\Gamma_1/N)$. Using this it is readily seen that the property to be virtually-$\mathcal{G}_\infty$ also enjoy these two properties.

Now using Wedderburn-Artin write $\mathbb{Q}G = \bigoplus_{i=1}^q \mathrm{M}_{n_i}(D_i)$ and take some order $\mathcal{O}_i$ in D_i for each i. By the aforementioned remark and [40, Lemma 4.6.9.] one has that $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_\infty$ if and only if $\prod_{i=1}^q \mathrm{GL}_{n_i}(\mathcal{O}_i)$ is. In light of the first paragraph of the proof of Lemma 6.5 we now see that (2) implies (1).

Suppose that $\mathcal{U}(\mathbb{Z}G)$ is virtually-$\mathcal{G}_\infty$ and let $H = \prod_{i=1}^m H_i \in \mathcal{G}_\infty$ (so $e(H_i) = \infty$ for all i) be a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$. To start:
Claim 1: G is a cut group, i.e. $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite and hence $\mathcal{Z}(\mathcal{O}_i)$ is finite for all i.

For this remark that if $e(\Gamma) = \infty$ for some finitely generated group Γ, then $\mathcal{Z}(\Gamma)$ is finite. Therefore also $\mathcal{Z}(H)$ is finite and hence $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ too[11]. The second part is well-known and is due to the fact that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) = \mathcal{U}(\mathcal{Z}(\mathbb{Z}G))$ and $\mathcal{Z}(\mathbb{Z}G)$ is an order in $\mathcal{Z}(\mathbb{Q}G) = \prod_{i=1}^q \mathcal{Z}(D_i)$. Hence one may use [40, Lemma 4.6.9.] to obtain that $\mathcal{U}(\mathcal{Z}(\mathcal{O}_i))$ is finite[12] for all i.

Next,
Claim 2: Let T be a finitely generated group with $e(T) = \infty$. If $P, Q \lneq T$ are normal

[10]This condition is not necessary, i.e the number of ends of $GL_2(\mathcal{O})$ for $\mathcal{O}$ an order in $\left(\frac{-2,-5}{\mathbb{Q}}\right)$ is not infinite. However including this case would make the proof unnecessarily lengthy.

[11]The subgroup $H \cap \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \leq \mathcal{Z}(H)$ is of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$.

[12]By Dirichlet's theorem this exactly means that $\mathcal{Z}(D_i)$ is $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$.

finitely generated subgroups such that $|P \cap Q| < \infty$ and PQ of finite index, then P or Q is finite.

Suppose such would exists. Then $e(PQ) = \infty$. Since by assumption $P \times Q \cong PQ/(P \cap Q)$ is commensurable with PQ also $e(P \times Q) = \infty$. However, the Cayley graph of a direct product is the cartesian product of the Cayley graphs. Using this one can see that the number of ends of a direct product of finitely generated groups is always one if P and Q are infinite, a contradiction.

Claim 3: $e(\mathrm{SL}_{n_j}(\mathcal{O}_j)) = \infty$ for all j such that $M_{n_j}(D_j)$ is different of a field or totally definite quaternion algebra (e.g. all j for which $n_j \geq 2$).

Denote $S_j := \mathrm{SL}_{n_j}(\mathcal{O}_j) \cap H$ which is of finite index in $\mathrm{SL}_{n_j}(\mathcal{O}_j)$, hence it is enough to proof that $e(S_j) = \infty$. Let p_k be the projection of H on H_k. Fix some j as in the claim. The condition is equivalent with saying that $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ is infinite [47]. In particular there exists some k such that $p_k(S_j)$ is infinite[13]. For such k we will now prove that $|p_k(\prod_{i \neq j} S_i)| < \infty$. For this consider $S := S_j \times \prod_{i \neq j} S_i$ which by the first claim is of finite index in H. Therefore $p_k(S)$ is of finite index in H_k and hence $e(p_k(S)) = \infty$. However, $p_k(S_j)$ and $p_k(\prod_{i \neq j} S_i)$ are subgroups as in the second claim[14], yielding the desired. Indeed, the two subgroups clearly commute, are normal in $\pi_k(S)$ and $p_k(S_j) \cap p_k(\prod_{i \neq j} S_i) \subseteq \mathcal{Z}(p_k(S))$ which is finite since $p_k(S)$ has infinitely many ends.

Now consider the set $\mathcal{I}_j := \{k \mid |p_k(S_j)| < \infty\}$. From the previous it follows that if $k \in \{1, \ldots, q\} \setminus \mathcal{I}_j$, then $p_k(S_j)$ is of finite index in H_k. Hence $S_j/(S_j \cap \prod\limits_{i \in \mathcal{I}_j} H_i)$ is a subgroup of finite index in $\prod_{k \notin \mathcal{I}_j} H_k$. As the quotient was with a finite subgroup, we obtain that S_j is virtually-$\mathcal{G}_\infty$ and hence $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ also. However under the conditions above SL_1 is virtual indecomposable [48, Theorem 1]. Therefore $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ in fact is even virtually a group with infinitely many ends and so in fact $e(\mathrm{SL}_{n_j}(\mathcal{O}_j)) = \infty$, as claimed.

Altogether: Claim 1 says that G is a cut group and consequently $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ is of finite index in $\mathrm{GL}_{n_j}(\mathcal{O}_j)$ for all j. In particular $e(\mathrm{GL}_{n_j}(\mathcal{O}_j)) = \infty$ if $n_j \geq 2$. Now Lemma 6.5 imply that $n_j = 2$, i.e. no higher matrix algebras appear in the decomposition of FG. In such a case no $\left(\frac{-2,-5}{\mathbb{Q}}\right)$ component arises. Indeed, following [4, table appendix] such a component can only arise if $F = \mathbb{Q}$ and G maps onto one of the groups with SmallGroupID [40,3], [240,89] or [240,90]. But a direct verification, e.g. via the Wedderga package on GAP, shows that these groups all have higher matrix components.

Consequently, Lemma 6.5 says that all matrix components of FG must be isomorphic to $\mathrm{M}_2(\mathbb{Q})$ and in particular $F = \mathbb{Q}$ (as F is contained in the center of every simple component). Furthermore, by [4, Th. 2.10. & Prop. 6.11.], if $\mathbb{Q}Ge$ is a division algebra D for some primitive central idempotent e of $\mathbb{Q}G$ then D is $\mathbb{Q}(\sqrt{-d})$ with $d \in \mathbb{Z}_{\geq 0}$ or a totally definite quaternion algebra over $\mathbb{Q}$. In summary, we obtained that all components of $\mathbb{Q}G$ are of the desired form. Conversely if $\mathbb{Q}G$ has only such components it follows e.g. from Lemma 6.5 that $\mathcal{U}(\mathbb{Z}G)$ is virtually in $\mathcal{G}_\infty$. *This finishes the proof of the first equivalence.*

Next, that only the parameters $(-1,-1)$ and $(-1,-3)$ is due to [75, Theorem 11.5.14] saying that else $\mathcal{U}(\mathcal{O})$ is cyclic for any order in $\left(\frac{-a,-b}{\mathbb{Q}}\right)$. In those cases $Ge \leq \mathcal{U}(\mathbb{Z}Ge)$ would have an abelian $\mathbb{Q}$-span and thus $\mathbb{Q}Ge \neq \left(\frac{-a,-b}{\mathbb{Q}}\right)$, a contradiction.

For the last part, remark first that by the commensurability of unit groups of orders $e(\mathcal{U}(\mathbb{Z}G)) = e(\prod_{i=1}^{q} \mathrm{GL}_{n_i}(\mathcal{O}_i))$. However the Cayley graph of a direct product is the cartesian product of the Cayley graphs. Using this we see that $e(Q \times P) = 1$ for any finitely

[13] Otherwise S_j would be finite and hence also the overgroup of finite index $\mathrm{SL}_{n_j}(\mathcal{O}_j)$.

[14] Instead of claim 2 one could have used the well known [70, 4.A.6.3.] saying that infinite finitely generated normal subgroups of a group with infinitely many ends need to have finite index.

generated group P, Q. Therefore $e(\mathcal{U}(\mathbb{Z}G)) = \infty$ if and only if $e(\mathrm{GL}_{n_{i_0}}(\mathcal{O}_{i_0})) = \infty$ for exactly one i_0 and the other factors are finite. In light of Lemma 6.5 and [4, Th. 2.10.] this happens exactly when $\mathbb{Q}G$ has exactly one $\mathrm{M}_2(\mathbb{Q})$ component and all the others are $\mathbb{Q}, \mathbb{Q}(\sqrt{-d})$ or $\left(\frac{-a,-b}{\mathbb{Q}}\right)$. Since $\mathrm{GL}_2(\mathbb{Z})$ is virtually free we see that in those cases $\mathcal{U}(\mathbb{Z}G)$ is indeed virtually free.

It remains to prove that the only finite groups for which this happens are D_6, D_8, Dic_3 and $C_4 \rtimes C_4$. Recall that the unit group of the maximal orders of $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ and $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ are respectively $\mathrm{SL}(2,3) \cong Q_8 \rtimes C_3$ and Dic_3. Thus by the work done till now we already know that $\mathcal{U}(\mathbb{Z}G)$ is a subgroup of finite index in $(D_8 \times U) *_{C_2 \times C_2 * U} (D_{12} \times U)$ where $U = A \times \mathrm{SL}(2,3)^s \times Dic_3^t$ for some s, t and with A abelian with $\exp(A) \mid 4, 6$. Using the description of torsion subgroups in amalgamated products we know that, up to conjugation, G is a subgroup of $C_2 \times C_2 * U$ or its contains transversal elements in one of the factors (i.e. D_8 or D_{12}). First suppose G is conjugated to a subgroup of U. Recall that all subgroups of Dic_3 are cyclic and the only non-cyclic one $\mathrm{SL}(2,3)$ is Q_8. As $\mathbb{Q}[\mathrm{SL}(2,3)]$ has a component $\mathrm{M}_3(\mathbb{Q})$ one can conclude that the only way to have exactly matrix component, which moreover is $\mathrm{M}_2(\mathbb{Q})$, is for G to be Dic_3. No suppose G is not conjugated to a subgroup of the amalgamated part. Then we know from Proposition 2.7 that $G \setminus (G \cap (C_2 \times C_2 \times U))$ embeds in $\mathrm{GL}_2(Z)$. If G contains no amalgamated element, then G embeds it needs to be isomorphic to D_6 or D_8 (as D_{12} has two matrix components). In general since $G \cap (C_2 \times C_2 \times U)$ will be a strict subgroup it will be central. Moreover, in order to have not more matrix components, the intersection clearly has to be a central subgroup of order 2. Thus G is a central extension of D_6 or D_8 with a C_2. A look at the groups of order 12 and 16 tells us that G is isomorphic to either Dic_3 or $C_4 \times C_4$, finishing the proof. □

In upcoming work applications of Theorem 6.2 to the "blockwise Zassenhaus conjecture" will be investigated. In other words applications to the question whether He is conjugated inside $\mathcal{U}(\mathbb{Z}Ge)$ to a subgroup of Ge for any finite subgroup H of $V(\mathbb{Z}G)$.

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