A BASIS AND SCHUR–WEYL DUALITY FOR THE LOOP HECKE ALGEBRA

GEOFFREY JANSSENS, ABEL LACABANNE, LÉO SCHELSTRAETE AND PEDRO VAZ

ABSTRACT. The loop Hecke algebra is a generalization of the Hecke algebra to the loop braid group, introduced by Damiani, Martin and Rowell. We give a new presentation of the loop Hecke algebra provided a mild condition on the parameter and give a basis. We use higher rewriting theory to show linear independence and the combinatorics of Dyck paths to compute the cardinality of the basis. This proves a conjecture of Damiani–Martin–Rowel. We also give a representation theoretic interpretation of the loop Hecke algebra in terms of (non-semisimple) Schur–Weyl duality involving the negative half of quantum $\mathfrak{gl}(1|1)$.

CONTENTS

2020 *Mathematics Subject Classification.* 20F36,16T99,20G42.

Key words and phrases. loop braid groups, loop Hecke algebra, Schur–Weyl duality, indecomposable representations, R-matrix.

1. INTRODUCTION

The classical braid group B_n can be identified with the group of motions of n points in the plane $\mathbb{R}^2$. In a similar spirit, the *loop braid group* LB_n is the group of motions of n unlinked circles in the space $\mathbb{R}^3$. This definition was given by Dahm in his unpublished PhD thesis [10], then published and extended by Goldsmith [14]. The terminology is due to Baez, Wise and Crans [1], inspired by physical motivations. As its classical counterpart, the loop braid group admits many different definitions, reflecting its connections with various fields: as certain automorphisms of the free group on n generators [22, 31]; as certain braid-like objects called *welded braids* [13], in connection to virtual knot theory [18] and knotted surfaces [30]; or as the configuration space of n unlinked circles in $\mathbb{R}^3$ [4]. We refer the reader to [11] for an overview.

The loop braid group admits an Artin-like presentation (see e.g. [13]):

$$\mathrm{LB}_n := \left\langle \begin{array}{c} \sigma_1, \dots, \sigma_{n-1}, \\ \rho_1, \dots, \rho_{n-1} \end{array} \;\middle|\; \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1},\ \rho_i^2 = 1, \\ \rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1},\ \sigma_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\sigma_{i+1}, \\ \sigma_i\sigma_j = \sigma_j\sigma_i,\ \rho_i\rho_j = \rho_j\rho_i,\ \sigma_i\rho_j = \rho_j\sigma_i \text{ for } |i-j| > 1 \end{array} \right\rangle \tag{1}$$

The generators σ_i correspond to the ith circle passing through the $(i+1)$th circle; they generate a copy of the braid group $\mathrm{B}_n \hookrightarrow \mathrm{LB}_n$. The generators ρ_i correspond to permuting the ith circle and the $(i+1)$th circle; they generate a copy of the symmetric group $\mathrm{S}_n \hookrightarrow \mathrm{LB}_n$. The remaining relations are mixed relations, capturing how the copy of the braid group B_n and the copy of the symmetric group S_n interact inside LB_n.

To study a group, one looks for interesting representations. As the braid group B_n sits inside the loop braid group LB_n, a natural approach is to try to extend a representation of B_n to a representation of LB_n. Arguably, the most classical representation of B_n is the Burau representation [7]:

$$\mathbf{F}_n \colon \mathrm{LB}_n \to V^{\otimes n},$$

where V is a 2-dimensional complex vector space. It has a long history, with connection to the Alexander polynomial and a still-standing faithfulness conjecture for $n = 4$. It is known to factor through the Hecke algebra $\mathcal{H}_n$, defined as a quotient of the group algebra $\mathbb{C}[\mathrm{B}_n]$ by quadratic relations $\sigma_i^2 = (t+1)\sigma_i + t$.

Recently, Damiani, Martin and Rowell [12] introduced the *loop Hecke algebra* $\mathcal{LH}_n$ as an analogue for LB_n of the Hecke algebra[1]. This builds on earlier work by Vershinin [35] extending the Burau representation to the loop braid group. Their definition is an analogue of the definition of the Hecke algebra: it is a quotient of the group algebra $\mathbb{Z}[\mathrm{LB}_n]$ by certain quadratic relations (see Definition 1.2). Although not clear from the definition, it was shown in [12, Corollary 3.5] that $\mathcal{LH}_n$ is finite dimensional. Furthermore, rather surprisingly, the dimension should be independent of t under the intriguing condition that $t^2 \neq 1$:

Conjecture 1.1 (Damiani–Martin–Rowell)**.** *Let $\mathbb{C}_z$ be the complex numbers $\mathbb{C}$ seen as a left $\mathbb{Z}[t]$-module by evaluating t at $z \in \mathbb{C}$. Then for $z \neq \pm 1$:*

$$\dim_{\mathbb{C}} \mathcal{LH}_n \otimes_{\mathbb{Z}[t]} \mathbb{C}_z = \frac{1}{2}\binom{2n}{n}.$$

[1]To the authors' knowledge, there is no relationship with loop algebras as appearing in affine Lie theory.

The first aim of this paper is to confirm Conjecture 1.1. The main idea is a new presentation of $\mathcal{LH}_n$, valid when $z \neq \pm 1$, which does not explicitly depend on the parameter t. Using this new presentation, we are able to describe an explicit basis. Linear independence is shown in Section 2 using a higher analogue of the diamond lemma. We count the cardinality of the basis using the combinatorics of Dyck paths in Section 3. Finally, we show in Section 7 that the two presentations are equivalent when $z \neq \pm 1$, leading to a proof of the conjecture. A more detailed introduction is given in Section 1.1 and Section 1.2 below.

The second aim of this paper is to give a representation theoretic interpretation of the loop Hecke algebra. The braid group is Schur–Weyl dual to the quantum $U_q(\mathfrak{gl}(1|1))$ through the Burau representation [17]. That is, one can realize V as a representation of $U_q(\mathfrak{gl}(1|1))$, such that the braiding on $V^{\otimes n}$ induced by $\mathbf{F}_n$ coincides with the braiding induced by the quantum group $U_q(\mathfrak{gl}(1|1))$. Furthermore, the Hecke algebra fully describes $U_q(\mathfrak{gl}(1|1))$-intertwiners, in the sense that the algebra morphism

$$\mathcal{H}_n \otimes_{\mathbb{Z}[t]} \mathbb{C}(t) \to \mathrm{End}_{U_q(\mathfrak{gl}(1|1))}(V^{\otimes n})$$

is surjective. In this paper, we show that a similar statement holds for the loop Hecke algebra $\mathcal{LH}_n$. Since $\mathcal{LH}_n$ is "larger" than $\mathcal{H}_n$, we must "shrink" $U_q(\mathfrak{gl}(1|1))$. It turns out that the right answer is to consider its negative-half $U_q(\mathfrak{gl}(1|1))^{\leq 0}$. In Section 4 we recall some background on $U_q(\mathfrak{gl}(1|1))$ and its representations. Section 5 is the core of this second part of the paper, showing a Schur–Weyl duality between LB_n and $U_q(\mathfrak{gl}(1|1))^{\leq 0}$. Finally, we use this result in Section 6 to further study the loop Hecke algebra from a ring-theoretic perspective. A more detailed introduction is given in Section 1.3.

We now give a more detailed account of the main results and objects. We conclude with some further directions of research in Section 1.4.

Acknowledgment. G.J. is grateful to Fonds Wetenschappelijk Onderzoek Vlaanderen - FWO (grant 88258), and le Fonds de la Recherche Scientifique - FNRS (grant 1.B.239.22) for financial support. L.S. is supported by the Max Planck Institute for Mathematics (Bonn, Germany). The authors were supported by a PHC Tournesol Wallonie Bruxelles grant.

G.J. warmly thanks Celeste Damiani and Eric Rowell for explaining their work and telling about Conjecture 1.1 during the problem session of the Banff workshop "Skew Braces, Braids and the Yang-Baxter Equation" (24w5201).

1.1. **A parameter independent presentation of the loop Hecke algebra.** We recall the definition of the loop Hecke algebra:

Definition 1.2 ([12, section 3B])**.** The *loop Hecke algebra* $\mathcal{LH}_n$ is the associative unital $\mathbb{Z}[t]$-algebra generated by $\sigma_1, \dots, \sigma_{n-1}$ and $\rho_1, \dots, \rho_{n-1}$, subject to the loop braid relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad \rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}, \tag{2}$$
$$\rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1}, \qquad \sigma_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\sigma_{i+1}, \tag{3}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad \rho_i\rho_j = \rho_j\rho_i, \qquad \sigma_i\rho_j = \rho_j\sigma_i, \qquad \text{for } |i-j| > 1, \tag{4}$$

and the quadratic relations

$$\rho_i^2 = 1, \qquad (\sigma_i - 1)(\sigma_i + t) = 0, \tag{5}$$

$$(\rho_i - 1)(\sigma_i + t) = 0, \qquad (\sigma_i - 1)(\rho_i + 1) = 0. \tag{6}$$

Note that relations (2), (3) and (4) are the defining relations of the loop braid group LB_n given in (1).

Remark 1.3. In [12] the loop Hecke algebra is defined as the $\mathbb{Z}[t, t^{-1}]$-algebra generated by the relations (2)-(6). However, as pointed out around [12, eq. (3-14)], it is senseful to define it over $\mathbb{Z}[t]$. The extension $\mathcal{LH}_n \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t^{\pm 1}]$ has the advantage that $t^{-1}(\sigma_i + t - 1)$ is an inverse for σ_i; however, we will not require this fact.

When $t - 1$ is invertible one could consider the following alternative generating set for $\mathcal{LH}_n$:

$$D_i = (\sigma_i - \rho_i)/(1 - t) \text{ and } U_i = (\sigma_i - t\rho_i)/(1 - t),$$

for $1 \leq i \leq n-1$. It turns out that for these generators $\mathcal{LH}_n$ has a more symmetric presentation, under some further conditions on the parameter. This and some experiments for small n using MAGMA motivate considering the following $\mathbb{Z}$-algebra:

Definition 1.4. For each $n \in \mathbb{N}_{>0}$, the *integral form of the loop Hecke algebra* $\widetilde{\mathcal{LH}}_n$ is the $\mathbb{Z}$-algebra with generators D_i and U_i for each $1 \leq i \leq n-1$, subject to the following relations[2]:

$$D_iD_i = D_i \qquad D_iU_i = 0 \qquad U_iD_i = U_i + D_i - 1 \qquad U_iU_i = U_i \tag{7}$$

for $1 \leq i \leq n-1$ and

$$D_iU_{i+1} = U_{i+1}D_i \qquad U_iD_{i+1} = 0 \qquad D_{i+1}U_i = D_{i+1} + U_i - 1 \tag{8}$$

$$D_iD_{i+1}D_i = D_{i+1}D_i = D_{i+1}D_iD_{i+1} \tag{9}$$

$$U_iU_{i+1}U_i = U_{i+1}U_i = U_{i+1}U_iU_{i+1} \tag{10}$$

for $1 \leq i \leq n-2$ and

$$U_iD_j = D_jU_i \qquad U_iU_j = U_jU_i \qquad D_iD_j = D_jD_i \qquad |i - j| > 1 \tag{11}$$

for $1 \leq i, j \leq n-1$.

Note that the U_i's and D_i's still satisfy the braid relations (9)-(10), but the quadratic relations (5)-(6) combine to the nicer relations (7), which do not involve the parameter t. Note also that the generators of $\widetilde{\mathcal{LH}}_n$ are idempotent elements.

An important step towards confirming Conjecture 1.1 is to prove that the presentation in Definition 1.4 is also one of the loop Hecke algebra.

Theorem A (Corollary 7.2)**.** Let $\mathbb{C}_z$ be the complex numbers $\mathbb{C}$ seen as a left $\mathbb{Z}[t]$-module by evaluating t at $z \in \mathbb{C}$. Then for $z \neq \pm 1$, the loop Hecke algebra $\mathcal{LH}_n$ (Definition 1.2) and its integral form $\widetilde{\mathcal{LH}}_n$ (Definition 1.4) are isomorphic over $\mathbb{C}$:

$$\mathcal{LH}_n \otimes_{\mathbb{Z}[t]} \mathbb{C}_z \cong \widetilde{\mathcal{LH}}_n \otimes_{\mathbb{Z}} \mathbb{C}.$$

[2]In fact, the relations $U_iU_{i+1}U_i = U_{i+1}U_i$ and $D_{i+1}D_iD_{i+1} = D_{i+1}D_i$ are consequences of the other relations; see Remark 7.4.

A more general statement is given in Theorem 7.1. The main difficulty in the proof is to verify that the image in $\mathcal{LH}_n$ of the relations (8)-(10) holds. The full Section 7.1 will be dedicated to that.

1.2. **A basis for the loop Hecke algebra and consequences.** The relations of $\widetilde{\mathcal{LH}_n}$ have the advantage of saying how to swap any two generators. This allows us to obtain a basis, which we now introduce. First, recall that a permutation $\tau \in \mathrm{S}_m$ is called 321-*avoiding* if there is no $i < j < k$ such that $\tau(i) > \tau(j) > \tau(k)$.

Definition 1.5. A word in the alphabet $\{U_i, D_i\}_{1 \leq i < n}$ is said to be $\widetilde{\mathcal{LH}_n}$-*reduced* if it is of the form

$$\omega = \underline{D}\,\underline{U}$$

for $\underline{D}$ (resp. $\underline{U}$) a 321-avoiding reduced word in the alphabet $\{D_i\}_{1 \leq i < n}$ (resp. $\{U_i\}_{1 \leq i < n}$) for each $1 \leq i < n$,[3] if D_i is a letter of $\underline{D}$, then U_i and U_{i-1} are *not* letters of $\underline{U}$:

$$D_i \in \underline{D} \Rightarrow U_i, U_{i-1} \notin \underline{U}. \tag{12}$$

We write $\mathrm{Red}(\widetilde{\mathcal{LH}_n})$ for the set of $\widetilde{\mathcal{LH}_n}$-reduced words.

Theorem B (Theorem 2.1)**.** For each $n \in \mathbb{N}_{>0}$, the set $\mathrm{Red}(\widetilde{\mathcal{LH}_n})$ of $\widetilde{\mathcal{LH}_n}$-reduced words defines a basis of $\widetilde{\mathcal{LH}_n}$.

We provide two proofs of linear independence of $\mathrm{Red}(\widetilde{\mathcal{LH}_n})$. The first one, given in in Section 2, works over $\mathbb{Z}$ and uses *higher linear rewriting theory* [32]. Rewriting theory is also known as *Gröbner bases theory* [5, 33] or *Bergman's diamond lemma* [3] depending on the context and the perspective. Readers familiar with the latter should be able to follow Section 2 without prior knowledge of [32]. In fact, our proof provides a "higher Gröbner basis" for $\widetilde{\mathcal{LH}_n}$, i.e. a solution to the word problem (see Corollary 2.5).

The second one works over $\mathbb{C}_z$, and is a by-product of the proof in Section 5 of our Schur–Weyl type theorem, see Remark 5.10.

Next, we count $\widetilde{\mathcal{LH}_n}$-reduced words:

Theorem C (Theorem 3.1)**.** The cardinality of $\mathrm{Red}(\widetilde{\mathcal{LH}_n})$ is $\frac{1}{2}\binom{2n}{n}$.

The proof is the content of Section 3. It uses the combinatorics of Dyck paths, which may be of independent interest.

Taken together, Theorem A, Theorem B and Theorem C imply that Conjecture 1.1 indeed holds.

Corollary D. Let $\mathbb{C}_z$ be the complex numbers $\mathbb{C}$ seen as a left $\mathbb{Z}[t]$-module by evaluating t at $z \in \mathbb{C}$. Then for $z \neq \pm 1$:

$$\dim_{\mathbb{C}} \mathcal{LH}_n \otimes_{\mathbb{Z}[t]} \mathbb{C}_z = \frac{1}{2}\binom{2n}{n}.$$

A more general statement is given in Corollary 7.3.

[3] That is, $\underline{D}$ and $\underline{U}$ are 321-avoiding reduced words each in their own alphabet.

In [12, Theorem 5.2] it is shown that the map defined by

$$\mathbf{F}_n\colon \mathrm{LB}_n \to \mathrm{End}(W^{\otimes n})\colon \begin{cases} \sigma_i \mapsto \mathrm{id}_W^{\otimes(i-1)} \otimes M'(\sigma_i) \otimes \mathrm{id}_W^{\otimes(n-i-1)} \\ \rho_i \mapsto \mathrm{id}_W^{\otimes(i-1)} \otimes M'(\rho_i) \otimes \mathrm{id}_W^{\otimes(n-i-1)} \end{cases} \tag{13}$$

with W a 2-dimensional vector space,

$$M'(\sigma_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix} \text{ and } M'(\rho_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is a representation[4] of the loop braid group LB_n which factors through the loop Hecke algebra $\mathcal{LH}_n$. The authors call $\mathbf{F}_n$ the extended super representation, denoted SP in [12], or the Burau–Rittenberg representation. Furthermore, they study the k-algebra $SP_n := \mathbf{F}_n(k\mathrm{LB}_n)$, where k is a field with a fixed element t. For instance, if $t \neq 1$, then $\dim_k SP_n = \frac{1}{2}\binom{2n}{n}$ [12, Theorem 5.8] and furthermore SP_{n-1} embeds in SP_n [12, Proposition 5.6]. Both properties however remained open for $\mathcal{LH}_n$.

Now concerning the former statement, comparing dimensions we see that $\mathcal{LH}_n \cong SP_n$ if $t \neq \pm 1$. We also obtain the latter from Theorem 3.1 as it yields that $\mathcal{LH}_n$ has a basis which is a subset of the basis of $\mathcal{LH}_{n+1}$. In summary:

Corollary E. For each $n \in \mathbb{N}_{>0}$ the following holds:

(1) the canonical morphism $\mathcal{LH}_n \to \mathcal{LH}_{n+1}$ is injective.
(2) the representation $\mathbf{F}_n$ induced on $\mathcal{LH}_n$ is faithful when $t \neq \pm 1$.

1.3. **On a Schur–Weyl duality for the loop Hecke algebra.** In the recent years there has been quite some interest in describing representations of LB_n which are extended from representations of the classical braid group B_n, e.g. [6, 9, 21]. There has been particular attention to so-called local representations which include those representations associated to a braided vector space. However, in contrast to the symmetric loop braid group[5], only a single R-matrix seems to be known that yields a local representation of LB_n, see [12, Remark 5.4]. The second main aim of this paper is to contribute to this and subsequently to use it to provide a Schur–Weyl duality for $\mathcal{LH}_n$..

(GJ: I will write the rest later.)

1.4. **Further questions.**

[4]Note that $M'(\sigma_i)$ is one form of the Burau representation. As pointed out in [12], if one considers another form of the Burau representation where the right down corner of $M'(\sigma_i)$ (and $M'(\rho_i)$) is replaced by 1, then one *does not* obtain a representation of LB_n.

[5]This is LB_n modulo the relation $\rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1}$. It is also called the *unrestricted virtual braid group* in [19].

2. A BASIS FOR THE INTEGRAL FORM OF THE LOOP HECKE ALGEBRA

In this section, we prove a basis theorem for the integral form of the loop Hecke algebra:

Theorem 2.1. *For each $n \in \mathbb{N}_{>0}$, the set of $\widetilde{\mathcal{LH}}_n$-reduced words (see Definition 1.5) defines a basis of $\widetilde{\mathcal{LH}}_n$ (see Definition 1.4).*

It will be convenient to package the $\mathbb{Z}$-algebras $\widetilde{\mathcal{LH}}_n$ for each n into one $\mathbb{Z}$-linear monoidal category: we define the *loop Hecke category* in Section 2.1. Section 2.2 describes reduced words using pattern avoidance. These two sections are preliminaries for Section 2.3, where we prove Theorem 2.1 using higher linear rewriting theory. This gives an intrinsic proof of Theorem 2.1. Another proof of linear independence will be given in Section 5 (see Remark 5.10), using Schur–Weyl duality.

2.1. **The loop Hecke category.**

Definition 2.2. The *loop Hecke category* $\widetilde{\mathcal{LH}}$ is the $\mathbb{Z}$-linear monoidal category given by the following presentation:

- one generating object, so that $\mathrm{ob}(\widetilde{\mathcal{LH}}) \cong \mathbb{N}$;
- two generating morphisms

$$U\colon 2 \to 2 \quad \text{and} \quad D\colon 2 \to 2$$

- subject to the following relations, where we abuse notation in (15), (16) and write U for $U \otimes \mathrm{id}_1$, we write $U_+ \coloneqq \mathrm{id}_1 \otimes U$, and similarly for D and D_+:

$$DD = D \qquad DU = 0 \qquad UD = U + D - 1 \qquad UU = U \tag{14}$$

$$DU_+ = U_+D \qquad UD_+ = 0 \qquad D_+U = D_+ + U - 1 \tag{15}$$

$$DD_+D = D_+D = D_+DD_+ \qquad UU_+U = U_+U = U_+UU_+ \tag{16}$$

The hom-spaces of $\widetilde{\mathcal{LH}}$ recover the algebras $\widetilde{\mathcal{LH}}_n$:

$$\mathrm{Hom}_{\widetilde{\mathcal{LH}}}(n, n) = \widetilde{\mathcal{LH}}_n,$$

where the identification is given by

$$U_i = \mathrm{id}_{i-1} \otimes U \otimes \mathrm{id}_{n-i-1} \qquad \text{and} \qquad D_i = \mathrm{id}_{i-1} \otimes D \otimes \mathrm{id}_{n-i-1}\,.$$

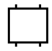
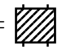
Note that the relations (11) are not explicitly part of the above presentation, as they correspond to the interchange law of a monoidal category. For that reason, we will refer to these relations as *interchange*, even when considered in the algebra $\widetilde{\mathcal{LH}}_n$.

2.1.1. *Symbolics.* In what follows, we will often abuse notation and write

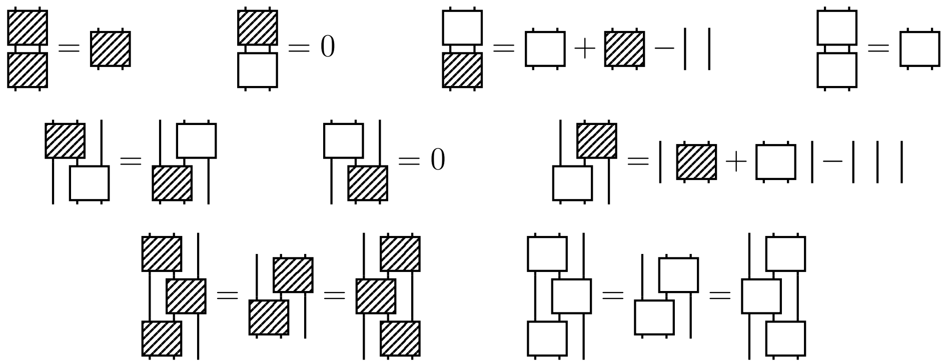
$$U = \mathrm{id}_{i-1} \otimes U \otimes \mathrm{id}_{n-i-1}, \quad U_+ \coloneqq \mathrm{id}_i \otimes U \otimes \mathrm{id}_{n-i-2}, \quad \text{and} \quad U_{++} \coloneqq \mathrm{id}_{i+1} \otimes U \otimes \mathrm{id}_{n-i-3}$$

for some i and n clear from context.

2.1.2. *Diagrammatics.* It will be convenient to have a diagrammatic notation for morphisms in the loop Hecke category. We define:

$U :=$  and $D :=$  .

In diagrammatic notation, the definition relations of $\widetilde{\mathcal{LH}}$ become



2.2. **A pattern-avoiding description of reduced words.** Recall from Definition 1.5 the notion of word and $\widetilde{\mathcal{LH}}_n$-reduced word. Say that a word is $\widetilde{\mathcal{LH}}$-*reduced* if it is $\widetilde{\mathcal{LH}}_n$-reduced for some $n \in \mathbb{N}$.

Recall that we call "interchange" the relations (11).

Lemma 2.3. *A word is $\widetilde{\mathcal{LH}}$-reduced if and only if it avoids the following patterns, up to interchange:*

$$UD \qquad U_+D \qquad UD_+ \tag{17}$$

$$DD \qquad UU \qquad DD_+D \qquad D_+DD_+ \qquad UU_+U \qquad U_+UU_+ \tag{18}$$

$$DU \qquad D_+U \qquad DU_+U \qquad D_+DU_+ \tag{19}$$

$$D_+DU_{++}U_+ \tag{20}$$

Here we use the abuse of notation from Section 2.1.1. For instance, fixing $n \in \mathbb{N}$, the pattern D_+U covers all patterns of the form $D_{i+1}U_i$ for $1 \leq i \leq n-2$.

Proof. For the purpose of the proof, we call R-*reduced*[6] any word that avoids patterns (17), (18), (19) and (20), up to interchange. It is clear that if a word is $\widetilde{\mathcal{LH}}_n$-reduced, then is it R-reduced. Let then ω be an R-reduced word. Since ω avoids the patterns (17), it is of the form $\omega = \underline{D}\ \underline{U}$ for $\underline{D}$ and $\underline{U}$ words in the alphabets $\{D_i\}_{1\leq i<n}$ and $\{U_i\}_{1\leq i<n}$, respectively. Since ω avoids the patterns (18), the words $\underline{D}$ and $\underline{U}$ are 321-avoiding and reduced.

It remains to check that ω verifies the condition (12). Let then $1 \leq i \leq n-1$ such that D_i appears as a letter in $\underline{D}$, and denote d the rightmost such letter in $\underline{D}$. Using interchange, move d to the right within $\underline{D}$, as much as possible. (In the process, other letters may move as well.) Since $\underline{D}$ is 321-avoiding and reduced, this expresses $\underline{D}$ as

$$\underline{D} = \underline{D}'\ D_iD_{i+1}\dots D_{i+m_+}D_{i-1}D_{i-2}\dots D_{i-m_-},$$

[6]The terminology "R-reduced" (or "R-normal forms") is a rewriting terminology, used explicitly in the proof of the basis theorem below. For the purpose of the proof, this can be taken as an ad-hoc terminology.

where D_i above is the chosen letter d, and m_+, m_- are non-negative integers. (If $m_- = 0$, then $D_{i-1}D_{i-2}\dots D_{i-m_-}$ is the empty word.) Note that if U_l is the leftmost letter of $\underline{U}$, we must have either $l < i - m_- - 1$ or $i + m_+ < l$, since $\omega = \underline{D}\ \underline{U}$ avoids the patterns (19).

Pick $1 \leq j \leq n-1$ such that U_j appears in $\underline{U}$, and let u be the leftmost such letter in $\underline{U}$. Similarly as above, we can express $\underline{U}$ as

$$\underline{U} = U_{j-n_-}\dots U_{j-2}U_{j-1}U_{j+n_+}\dots U_{j+1}U_j\ \underline{U}'$$

where U_j above is the chosen letter u, and n_-, n_+ are non-negative integers. If D_k is the rightmost letter of $\underline{D}$ then we must have either $k < j - n_-$ or $j + n_+ + 1 < k$.

Let

$$M_\pm := \begin{cases} m_\pm & \text{if } m_\pm \geq 1, \\ -m_\mp & \text{otherwise} \end{cases} \quad \text{and} \quad N_\pm := \begin{cases} n_\pm & \text{if } n_\pm \geq 1, \\ -n_\mp & \text{otherwise} \end{cases}.$$

The letters D_{i-M_-} and D_{i+M_+} (resp. U_{i-N_-} and U_{i+N_+}) are, up to interchange, rightmost letters in D (resp. leftmost letters in U). The conditions above give:

$$\begin{aligned} &(i - M_- < j - n_- && \text{or} && j + n_+ + 1 < i - M_-) \\ \text{and}\quad &(i + M_+ < j - n_- && \text{or} && j + n_+ + 1 < i + M_+) \\ \text{and}\quad &(j - N_- < i - m_- - 1 && \text{or} && i + m_+ < j - N_-) \\ \text{and}\quad &(j + N_+ < i - m_- - 1 && \text{or} && i + m_+ < j + N_+). \end{aligned}$$

Using that $M_\pm \leq m_\pm$ and $N_\pm \leq n_\pm$, we see that $i - M_- < j - n_-$ and $j - N_- < i - m_- - 1$ cannot hold at the same time; similarly, $j + n_+ + 1 < i + M_+$ and $i + m_+ < j + N_+$ cannot hold at the same time. It follows that:

$$\begin{aligned} &(j + n_+ + 1 < i - M_- && \text{or} && i + m_+ < j - N_-) \\ \text{and}\quad &(i + M_+ < j - n_- && \text{or} && j + N_+ < i - m_- - 1). \end{aligned}$$

If the first equation holds and $M_- \geq 0$, then $j + 1 < i$ and the pair (i, j) verifies condition (12); similar statements hold for the three other inequalities. Moreover, at least one element of $\{M_-, M_+\}$ is non-negative; and similarly for $\{N_-, N_+\}$. Hence, it only remains to check the following two cases:

- $M_-, N_+ < 0$: this implies that $m_- = n_+ = 0$;
- $M_+, N_- < 0$: this implies that $m_+ = n_- = 0$.

In both cases, we can use the avoidance of pattern (20) to conclude that we have either $i < j$ or $j + 1 < i$. $\square$

2.3. **Proof of the basis theorem via rewriting theory.** We wish to show Theorem 2.1. Recall that given a linear monoidal category, a hom-basis is a basis for each hom-space. Theorem 2.1 equivalently states that $\widetilde{\mathcal{LH}}$-reduced words define a hom-basis of the loop Hecke category $\widetilde{\mathcal{LH}}$ (Definition 2.2).

With the pattern-avoidance description of reduced words given in Lemma 2.3, it is not difficult to show the following:

Lemma 2.4. *For each $n \in \mathbb{N}$, the set of $\widetilde{\mathcal{LH}}_n$-reduced words generates $\widetilde{\mathcal{LH}}_n$ as a $\mathbb{Z}$-module.*

Proof. Let ω be any word in the alphabet $\{U_i, D_i\}_{1\leq i<n}$. Each pattern described in Lemma 2.3 can be rewritten using a relation in $\widetilde{\mathcal{LH}}_n$. This follows from the defining relations and the relations

$$D_+DU_+ = 0, \quad DU_+U = 0 \quad \text{and} \quad D_+DU_{++}U_+ = 0,$$

which are easy consequences of the defining relations. As long as ω contains one of the patterns in Lemma 2.3, we continue rewriting it. This process will eventually terminate, as rewriting a pattern strictly decreases the number of letters. This shows that ω can be expressed as a linear combination of reduced words, using relations in $\widetilde{\mathcal{LH}}_n$. □

To show linear independence, we use *higher linear rewriting theory*, as introduced in [32]. In fact, the case of the loop Hecke category is rather simple compared to other monoidal categories, and combining linear rewriting (see [15]) and higher rewriting (see e.g. [16]) is relatively straightfoward; see Remark 2.11 for a discussion. In particular, it allows us to phrase our discussion and review of [32] in terms that resemble the classical theory of Gröbner bases [5, 33], or Bergman's diamond lemma [3], for associative algebras. We begin with an informal discussion of the main ideas; they are (semi-)formalized in Section 2.3.1, which gives a minimal review of the relevant theory from [32]. Section 2.3.2 explains how the theory is applied to the loop Hecke category, the full details being postponed to Appendix A.

The proof of Lemma 2.4 defined a process that rewrites every word as a linear combination of reduced words; it is encapsulated in Figure 1 (symbolics) and Figure 2 (diagrammatics). The idea of rewriting theory is to formalize this process as an algorithm, where each step is called a *rewriting step*; by studying the properties of this algorithm, we will deduce linear independence. More precisely, we wish to show that not only can we rewrite a word as a linear combination of reduced words, but moreover this linear combination is *unique*. The latter is highly non-obvious, since a word can have two forbidden patterns at the same time, and our process does not choose which one should be rewritten first (in other words, the algorithm it defines is not deterministic). For instance the word DU_+D can be rewritten in two different ways:

$$\begin{tikzcd}[row sep=small] & D_+U + D_+D - D_+ \\ D_+UD \arrow[ur] \arrow[dr] & \\ & D_+D + UD - D \end{tikzcd} \tag{21}$$

Such a pair of rewriting steps is called a (local) *branching*. While the two rewriting steps have distinct target, one can check that they *confluate*, it the sense that one can use further rewriting steps to reach a common target:

$$\begin{tikzcd}[row sep=small] & D_+U + D_+D - D_+ \arrow[r] & D_+ + U - 1 + D_+D - D_+ \arrow[dr, equal] & \\ D_+UD \arrow[ur] \arrow[dr] & & & D_+D + U - 1 \\ & D_+D + UD - D \arrow[r] & D_+D + U + D - 1 - D \arrow[ur, equal] & \end{tikzcd}$$

We say that the algorithm *confluates* if every branching confluates; in this case, a word always rewrites as a linear combination of reduced words *in a unique way*. In fact, to show confluence

$$DD \to D \qquad DU \to 0 \qquad UU \to U \qquad UD \to U + D - 1$$

same-label rewriting steps

$$U_+D \to DU_+ \qquad UD_+ \to 0 \qquad D_+U \to D_+ + U - 1$$
$$DD_+D \to D_+D \qquad D_+DD_+ \to D_+D$$
$$UU_+U \to U_+U \qquad U_+UU_+ \to U_+U$$

distinct-label rewriting steps

$$D_+DU_+ \to 0 \qquad DU_+U \to 0 \qquad D_+DU_{++}U_+ \to 0$$

additional rewriting steps

Figure 1: A higher Gröbner basis for the loop Hecke category.

it suffices to show confluence of branchings that "overlap"; they are called *critical branchings*. For instance, the branching in (21) is a critical branching. If all critical branchings confluate, we say that the algorithm *critically confluates*.

Linear combinations on which the algorithm terminates are called *normal forms* (or *reduced*); if a word is a normal form, it is called a *monomial normal form*. In our setting, monomial normal forms are precisely $\widetilde{\mathcal{LH}}$-reduced words; this is essentially the content of Lemma 2.3. To sum up:

> SLOGAN (SEE THEOREM 2.10): If the algorithm in Figure 1 terminates and critically confluates, then $\widetilde{\mathcal{LH}}$-reduced words define a hom-basis of the loop Hecke category $\widetilde{\mathcal{LH}}$, showing Theorem 2.1.

As a byproduct, we get a solution to the word problem; that is, an algorithm that decides whether two (linear combination of) words are equal in $\widetilde{\mathcal{LH}}$. This is the perspective of Gröbner bases. For that reason, the oriented relations underpinning the algorithm may be called a *higher Gröbner basis*; that is, a Gröbner basis for a linear monoidal category.

In the terminology defined in Section 2.3.1:

Corollary 2.5. *The higher linear rewriting system given in Figure 1 is a higher Gröbner basis for the loop Hecke category.*

2.3.1. *A minimal review of higher linear rewriting theory.* We give a minimal review of [32] suitable for our purpose. In the interest of length, we sometimes stay at a semi-formal level of explanation. Experts are referred to Remark 2.11 for comparison with [32].

Fix k a commutative ring. Let

$$\mathcal{C} = \langle \mathsf{X}_0 \mid \mathsf{X}_1 \mid \mathsf{R} \rangle_{\otimes,k}$$

be a presented linear monoidal category. This means that:

- X_0 is the set of generating objects. We write $\mathsf{X}_0^* = \mathrm{ob}(\mathcal{C})$ the set of objects generated by X_0 under the tensor product $\otimes$.

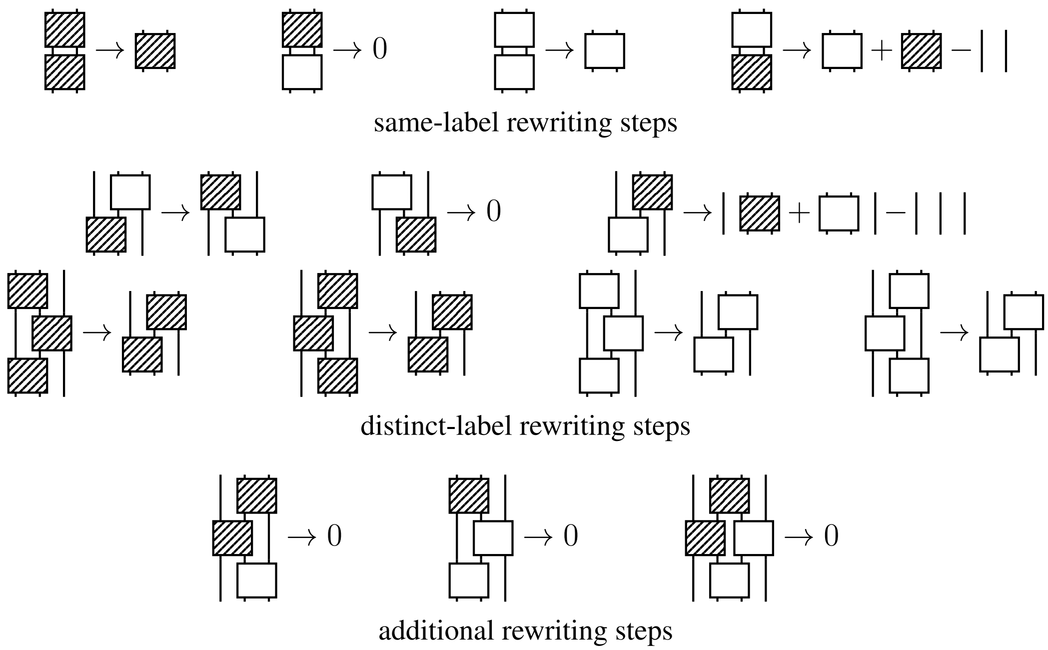


Figure 2: The higher Gröbner basis from Figure 1 in diagrammatic notation.

- X_1 is the set of generating morphisms, or *generators*, equipped with source and target maps $s, t\colon \mathsf{X}_1 \to \mathsf{X}_0$. A generator $f \in \mathsf{X}_1$ can be "extended" by identities of objects, giving a *whiskered generator* $\mathrm{id}_a \otimes f \otimes \mathrm{id}_b$ for each $a, b \in \mathsf{X}_0^*$. The source and target maps extend to whiskered generators in the natural way, and whiskered generators with matching source and target can be composed.

  For each pair of objects (a, b), we write $\mathsf{X}^*(a, b)$ the set of compositions of whiskered generators with source a and target b, regarded up to the interchange law:

$$(f \otimes \mathrm{id}_{t(g)}) \circ (\mathrm{id}_{s(f)} \otimes g) = (\mathrm{id}_{t(f)} \otimes g) \circ (f \otimes \mathrm{id}_{s(g)}).$$

  An element of $\mathsf{X}^*(a, b)$ is called a *monomial*.

  For each pair of objects (a, b), we write $\mathsf{X}^l(a, b) := \langle \mathsf{X}^*(a, b) \rangle_k$, the free k-module generated by the set $\mathsf{X}^*(a, b)$. An element of $\mathsf{X}^l(a, b)$ is called a *vector*. We write X^* (resp. X^l) the union of all the $\mathsf{X}^*(a, b)$'s (resp. $\mathsf{X}^l(a, b)$'s).
- R is a subset $\mathsf{R} \subset \mathsf{X}^l(a, b)$.

In the case of the loop Hecke Category, we have $\mathsf{X}_0 = \{1\}$, $\mathsf{X}_0^* \cong \mathbb{N}$ and $\mathsf{X}_1 = \{D, U\}$. Monomials are the same as words in symbolic notation, or diagrams in diagrammatic notation.

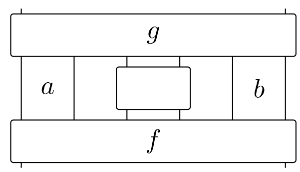
Definition 2.6. In the context of linear monoidal categories, a *higher linear rewriting system* is a presentation of a linear monoidal category, such that each relation comes equipped with an orientation, and the source of each relation is a monomial.

In our notation, this means that R is an abstract set equipped with source map $s\colon \mathsf{R} \to \mathsf{X}^*(a,b)$ and target map $s\colon \mathsf{R} \to \mathsf{X}^l(a,b)$. We often abuse notation by writing R for the data $(\mathsf{X}_0 \mid \mathsf{X}_1 \mid \mathsf{R})$, and abuse terminology by calling R a higher linear rewriting system.

Recall that a vector is a linear combination of monomials. For a vector $v \in \mathsf{X}^l$, we write $\operatorname{supp}(v)$ its *support*, that is, the set of monomials appearing in the linear decomposition of v.

Let $(\mathsf{X}_0 \mid \mathsf{X}_1 \mid \mathsf{R})$ be a higher linear rewriting system. We define:

- A *context* is a "monomial with a hole". In diagrammatic terms, a context Γ is:

$\Gamma =$ 

  where $a, b \in \operatorname{ob}(\mathcal{C})$ are objects and $f, g \in \mathsf{X}^*$ are monomials, suitably composable. Given a context Γ, we can *contextualize* a generating rewriting step $r\colon s(r) \to t(r)$ as $\Gamma[r]\colon \Gamma[s(r)] \to \Gamma[t(r)]$, provided source and target are compatible.
- A *rewriting step* is a rule of the form

$$\lambda\Gamma[r] + v\colon \lambda\Gamma[s(r)] + v \to \lambda\Gamma[t(r)] + v,$$

  where $\lambda \in k \setminus \{0\}$ is a non-zero scalar, Γ is a context, $r \in \mathsf{R}$ is a generating rewriting step and $v \in \mathsf{X}^l$ is a vector such that $\Gamma[s(r)] \notin \operatorname{supp}(v)$. Again, it is implicit that $\Gamma[r]$ and v have the same source and target.

  We say that $\lambda\Gamma[r]+v$ is a rewriting step *of type* r. We denote by R^+ the set of rewriting steps. Source and target maps naturally extend to R^+.

For instance, in our example $r = DD \to D$ is a generating rewriting step, $\Gamma = D_+[-]U$ is a context and $v = UD$, $v' = D_+DDU$ are vectors (in fact, monomials). We have that $\Gamma[r] = D_+DDU \to D_+DU$ is a rewriting step, viewed a contextualization of r. The rule

$$\Gamma[r] + v = D_+DDU + UD \to D_+DU + UD$$

is a rewriting step, but not the rule $\Gamma[r]+v' = D_+DDU + D_+DDU \to D_+DU + D_+DDU$, as it does not verify the condition on the support. This condition is known as the positivity condition, which explains the notation R^+.

Having defined rewriting steps, we have the following notions:

- A *rewriting sequence* is a finite sequence of rewriting steps $(\alpha_i)_{1\leq i\leq N}$ such that we have $s(\alpha_{i+1}) = t(\alpha_i)$;
- A *branching*[7] is a pair of rewriting steps (α, β) with the same source;
- A *confluence* is a pair of rewriting sequences (α', β') with the same target;
- A branching (α, β) is *confluent* (we say that it *confluates*) if it admits a confluence (α', β') such that $t(\alpha) = s(\alpha')$ and $t(\beta) = s(\beta')$.

There is an intuitive notion of the multiset of generators in a given monomial; for instance, the monomial $(D \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes D)$ has generators $\{D, D\}$. Given a rewriting step $\alpha = \Gamma[r]$ for $r \in \mathsf{R}$ a generating rewriting step, we call the multiset of generators in $s(r)$ the "generators

[7]More precisely, this is the definition of a *local* branching; we abuse terminology in this review.

associated to α"; they constitute the pattern on which we apply the rewriting rule. A branching (α, β) is *monomial* if its source $s(\alpha) = s(\beta)$ is a monomial. If further α and β have generators in common, we say that (α, β) is an *overlapping branching*. For instance, the branching given in (21) is an overlapping branching, as the two rewriting steps have the generator "U" in the middle of the monomial D_+UD in common.

Just like rewriting steps, a branching (α, β) can be contextualized as $(\Gamma[\alpha], \Gamma[\beta])$; we say that a branching is *minimal* if it is not the (non-trivial) contextualization of another branching.

Definition 2.7. In the context of linear monoidal categories, a *critical branching* is a minimal overlapping branching.

Definition 2.8. A higher linear rewriting system is said to *terminate* if there is no infinite sequence of rewriting steps, and to *critically confluate* if all critical branchings confluate.

Now we pause the review to emphasize a special feature of our setting. Note that for a generic higher linear rewriting system, *contextualization needs not be injective*. That is, if $f, g \in \mathsf{X}^*$ are monomials and Γ is a context, having $f \neq g$ does *not* imply that $\Gamma[f] \neq \Gamma[g]$[8]. This is because we consider monomials up to the interchange law, and as a context may connect two regions of a diagram, it may allow "floating morphisms" to move from one region to another. This fact is in contrast with the classical settings of linear rewriting in associative algebras or commutative algebras, where contexts *are* indeed injective. This makes the general theory of higher linear rewriting subtler than its classical counterpart; see [32].

However, in the case of the loop Hecke category, contexts *are* injective: if $f, g \in \mathsf{X}^*$ are monomials such that $f \neq g$ and Γ is a context, then $\Gamma[f] \neq \Gamma[g]$. This makes the theory similar to the classical setting, and one finds a statement analogous to (say) Bergman's diamond lemma.

Definition 2.9. In the context of higher linear rewriting system with injective contexts, A *higher Gröbner basis* is a higher linear rewriting system that terminates and critically confluates.

If $v \in \mathsf{X}^l$ is a vector such that no rewriting step has source v, we say that v is a *normal form*; if further $v \in \mathsf{X}^*$ is a monomial, then v is a *monomial normal form*.

Theorem 2.10. *In the context of higher linear rewriting system with injective contexts, If* R *is a higher Gröbner basis presenting a linear monoidal category* $\mathcal{C}$*, monomial normal form defines a hom-basis of* $\mathcal{C}$*.*

Remark 2.11 (comparison with [32]). In [32], the interchange law is made explicit as a modulo rule; in particular, X^* denotes the set formal compositions of whiskered generators, *not* regarded up to the interchange law. Also, the notion of higher Gröbner bases is only implicit in [32], and equivalent to the notion of a convergent higher linear rewriting system modulo interchangers.

To view Theorem 2.10 as a corollary of the results of [32], one requires a strongly compatible terminating order invariant under interchangers [32, Definition 3.58]. Since contexts are injective, an order is strongly compatible if and only if it is compatible, we can choose the smallest compatible order $\succ_{\mathsf{R}}$ for each linear rewriting system $\mathsf{R}(a, b)$ [32, Definition 3.48]. Thanks to

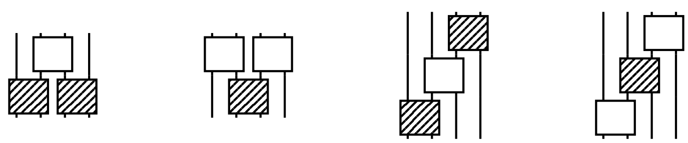
[8] Here the inequality is an inequality as monomials in the free monoidal category X^*, not an inequality as elements of the monoidal category presented by R.

[32, Lemma 3.41], termination implies that $\succ_{\mathsf{R}}$ is terminating. In other words, having injective contexts put us in essentially the same setting as associative algebras; see [15].

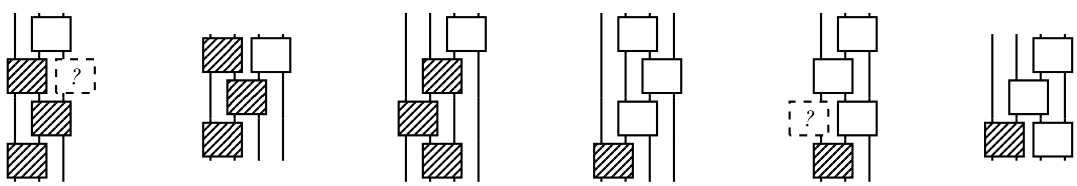
2.3.2. *A rewriting approach to the loop Hecke category.* Let R be the higher linear rewriting system defined in Figure 1. We have already argued that R terminates (see the proof of Lemma 2.4) and that monomial normal forms are precisely $\widetilde{\mathcal{LH}}$-reduced words (Lemma 2.3). To show Theorem 2.1 using Theorem 2.10, it only remains to show that R is critically confluent. This splits in two steps: enumerating all the critical branchings, and showing that they confluate. Both tasks are cumbersome, but apart from the subtlety of indexed branchings explained below, essentially straightforward.

Enumeration is best done diagrammatically: one tries to match patterns up to rectilinear isotopies. See Section 2.1.2 and Figure 2 for the diagrammatics. We illustrate the process with branchings (α, β) where α is of type $U_+D \to DU_+$ and β is of type one of the distinct-label rewriting steps. The full analysis is given in Appendix A.

Lemma 2.12. *The following is a complete list of critical branchings (α, β) where α is of type $U_+D \to DU_+$ and β is of type one of the distinct-label rewriting steps:*

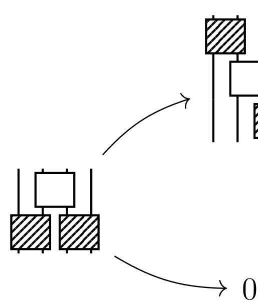


(22)



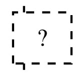
(23)

Here we describe a branching by its source, leaving its branches implicit. (Boxes with "?" will be explained below.) For instance, the first diagram encodes the following branching:



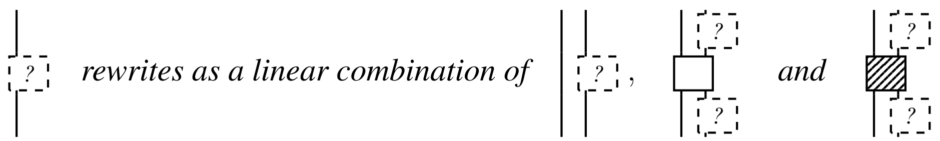
Here we remind the reader that we view diagrams modulo the interchange law, so that we can slide the Ds past each other to be able to apply each of the two rewriting steps. This branching is easily seen to confluate, as the top branch rewrites to zero using the rewriting step $UD_+ \to 0$.

Because we work modulo the interchange law, it may happen that an arbitrary diagram is "stuck" in between two rewriting rules. This happens for instance in the first branching of (23), where



denotes an arbitrary diagram. This is known as an *indexed branching* [16], a phenomenon typical of higher rewriting. A priori, this leads to an infinite family of branchings. However, one can always rewrite this diagram into a normal form. As we already know that normal forms are precisely the reduced words, it is not hard to check the following:

Lemma 2.13. *Denote an arbitrary diagram with a dashed box marked with "?", different boxes indicating (a priori) different diagrams. We have that:*



We have an analogous statement when flipping all diagrams along the vertical axis.

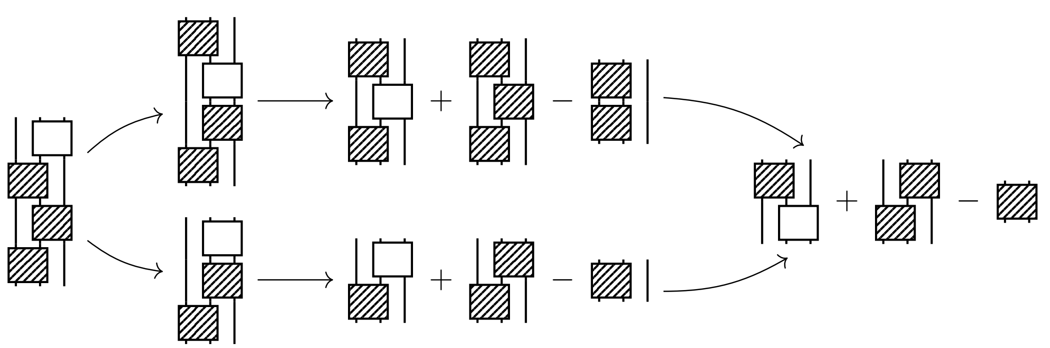
This reduces the analysis of indexed branchings to three cases:

(24)

Lemma 2.14. *The critical branchings of Lemma 2.12 are confluent.*

Proof. We have already seen that the first branching of (22) is confluent. The three other branchings of (22) are similar.

Consider the first branching of (23). As we argued above, it suffices to consider the three cases in (24). In fact, if = then both branches rewrite to zero using the rewriting step $UD_+ \to 0$. Moreover, if = then we can use the rewriting step $U_+D \to DU_+$ on both branches to slide this U downward, therefore reducing to the case = . In this latter case, we have:



The other indexed branching of (23) works similarly. The confluence of the remaining branchings is straightforward to check. □

3. Cardinality of reduced words and Dyck paths

In this section, we prove that the set of reduced words $\mathrm{Red}(\widetilde{\mathcal{LH}}_n)$ (see Definition 1.5), i.e. the basis obtained in Theorem 2.1, has the conjectured cardinality.

Theorem 3.1. *For each $n \in \mathbb{N}_{>0}$, the cardinality of $\mathrm{Red}(\widetilde{\mathcal{LH}}_n)$ is $\frac{1}{2}\binom{2n}{n}$.*

The proof of Theorem 3.1 will consist of two steps.

First, we rephrase the problem in terms of Dyck paths (Lemma 3.3). A *Dyck path of semilength* n is a path in the lattice $\mathbb{Z}^2$ between $(0,0)$ and (n,n), consisting only of steps $u := (0,1)$ (u-*step*) and $r := (1,0)$ (r-*step*), and such that the path always lies above[9] the diagonal $d = \{(x,y) \mid x = y\}$. A Dyck path can be encoded as words in u and r, reading from left to right as the path goes from $(0,0)$ to (n,n); see Figure 3 for an example.

We denote

$$\mathrm{Dyck}_n := \{\text{Dyck paths of semilength } n\}$$

and for a path $P \in \mathrm{Dyck}_n$ and $(a,b) \in \mathbb{Z}^2$ we write $(a,b) \in P$ whenever (a,b) lies on P.

In the first step we show that the $\widetilde{\mathcal{LH}}_n$-reduced words correspond to the following set.

Definition 3.2. For $n \in \mathbb{N}$:

$$\widetilde{\mathrm{Dyck}}_n := \big\{(P,Q) \in \mathrm{Dyck}_n \times \mathrm{Dyck}_n \mid (i,i) \notin P \Rightarrow (i,i) \text{ and } (i-1,i-1) \in Q\big\}.$$

The second step of the proof of Theorem 3.1 will consist of relating $\widetilde{\mathrm{Dyck}}_n$ to another set whose cardinality is easily seen to be the desired one, see Lemma 3.4 below.

Recall that $\mathrm{S}_n(321)$ denotes the set of 321-avoiding permutations in the symmetric group S_n and recall $\mathrm{Red}(\widetilde{\mathcal{LH}}_n) \subset \mathrm{S}_n(321) \times \mathrm{S}_n(321)$ from Definition 1.5. Both 321-avoiding permutations and Dyck paths count Catalan numbers. There are several bijections realizing that fact; for our purpose, we are interested in the one given by Mansour, Deng and Du [20].

Although we do not use them explicitly, the references [34, 8] have been helpful guides to the literature.

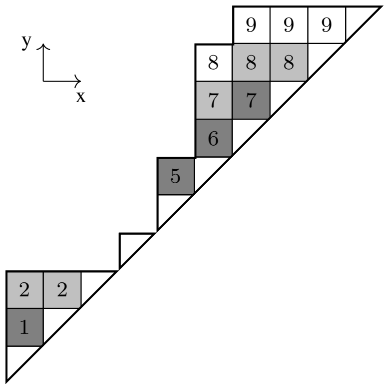
Lemma 3.3. *The Mansour–Deng–Du bijection $\mathrm{MDD}\colon \mathrm{Dyck}_n \to \mathrm{S}_n(321)$ between Dyck paths and 321-avoiding permutations is such that for $P \in \mathrm{Dyck}_n$, we have $(i,i) \notin P$ if and only if $s_i \in \mathrm{MDD}(P)$. In particular, it induces a bijection:*

$$\mathrm{Red}(\widetilde{\mathcal{LH}}_n) \cong \widetilde{\mathrm{Dyck}}_n.$$

Proof. We describe the bijection $\mathrm{MDD}\colon \mathrm{Dyck}_n \to \mathrm{S}_n(321)$, following [20, section 2.1]. An example of the procedure is given in Figure 3, following [20, Fig. 1].

Let $P \in \mathrm{Dyck}_n$ be a Dyck path. A *cell* is a size-one square in $\mathbb{Z}^2$ that lies between P and the diagonal $d = \{(x,y) \mid x = y\}$. We identify a cell with the coordinate (i,j) of its bottom-left corner. We label each cell with its y-coordinate j. A cell is *essential* if $(i,j-1) \to (i,j)$ and $(i,j) \to (i,j+1)$ are steps in P; in other words, if the point (i,j) lies between two "up" steps. Given an essential step (i,j), its *zigzag strip* is the set of cells adjacent to P between (i,j) and (n,n).

[9] One could equivalently define them to be paths below the diagonal d.



$$\xmapsto{\text{MDD}} (\sigma_1)(\sigma_2)(\sigma_7\sigma_6\sigma_5)(\sigma_8\sigma_7)(\sigma_9\sigma_8)$$

Figure 3: Example of the Mansour–Deng–Du's bijection, following [20, Fig. 1]. The Dyck path is $uuurrrruruuruuururrrr$. Each cell is equipped with its label, that is, its y-coordinate. Shadings emphasize the five zigzag strips.

Let S_n be the symmetric group on $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$. We associate an element $\mathrm{MDD}(P) \in \mathrm{S}_n$ to the Dyck path P, by induction on the number of essential cells:

- If P has no essential cell, then $P = (ru)^n$, and we associate the identity $1 \in \mathrm{S}_n$;
- Assume instead that P has essential cells. Let (i, j) be the rightmost essential cell in P and let Z be its zigzag strip. Let $\{i, i+1, \ldots, i+k\}$ be the set of labels of cells in Z and let $P' := P \setminus Z$ be the Dyck path obtained from P by removing the zigzag strip Z. Note that $\mathrm{MDD}(P') \in \mathrm{S}_n$ is defined by induction. We set $\mathrm{MDD}(P) = \mathrm{MDD}(P')(\sigma_{i+k} \ldots \sigma_{i+1}\sigma_i)$.

It was shown in [20, theorem 3] that this procedure defines a bijection between Dyck paths and 321-avoiding permutations. If P is a Dyck path, then $(i, i) \in P$ if and only if no cell of P has i as its y-coordinate, that is, if and only if $\sigma_i \notin \mathrm{MDD}(P)$. The lemma follows. □

The second and last step of the proof of Theorem 3.1 consists in relating $\widetilde{\mathrm{Dyck}}_n$ to another set with the expected cardinality:

Lemma 3.4. *Denote by* $\mathrm{Path}(n, n-1)$ *the set of paths from* $(0, 1)$ *to* (n, n) *consisting in steps* $r = (1, 0)$ *and* $u = (0, 1)$. *There is a bijection:*

$$\widetilde{\mathrm{Dyck}}_n \cong \mathrm{Path}(n, n-1).$$

Proof. For each pair $(P, Q) \in \widetilde{\mathrm{Dyck}}_n$, we think of P as sitting above the diagonal $d = \{(x, y) \mid x = y\}$, and Q as sitting below the diagonal. We begin with a few definitions:

- If P (resp. Q) is incident to the diagonal at points (i, i) and (j, j) for $0 \le i, j \le n$, we write $P_{i,j}$ (resp. $Q_{i,j}$) the sub-Dyck path of P (resp. Q) from (i, i) to (j, j);
- A *squiggly P-line* is a sub-Dyck path of P of the form $P_{i,i+k} = (ur)^k$ for $i \ge 1$;
- A *maximal squiggly P-line* is a squiggly P-line $P_{i,i+k}$ maximal with respect to k, that is, such that neither $P_{i-1,i-1+k}$ nor $P_{i,i+k+1}$ is a squiggly P-line;

We stress the condition $i \ge 1$ for a squiggly P-line: a squiggly P-line never contains the first u-step of P.

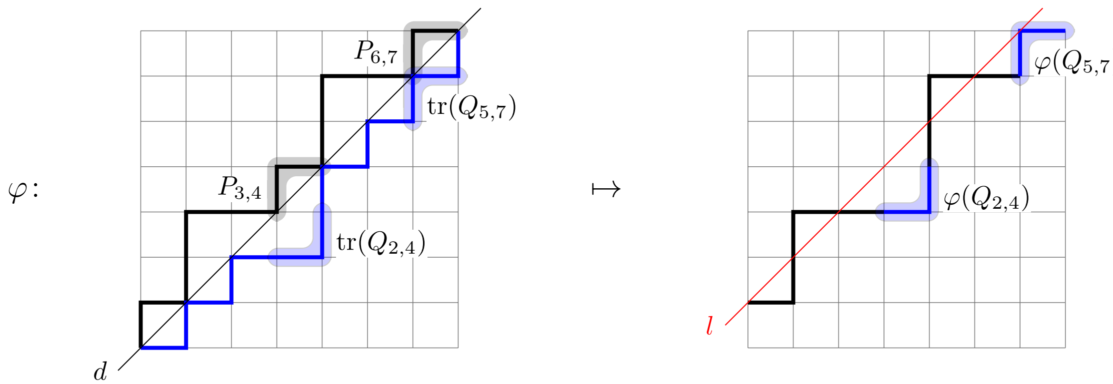


Figure 4: Definition of $\varphi(P,Q)$ for some pair of Dyck paths $(P,Q) \in \widetilde{\mathrm{Dyck}}_7$, with $P = uruurruruurrur$ depicted in black above the diagonal d and $Q = rururruururuu$ depicted in blue below the diagonal. The two maximal squiggly P-lines of P are $P_{3,4}$ and $P_{6,7}$, shaded in black; the associated sub-Dyck paths of Q are $Q_{2,4}$ and $Q_{5,7}$, respectively. Their truncations $\mathrm{tr}(Q_{2,4})$ and $\mathrm{tr}(Q_{5,7})$ are shaded in blue. To obtain $\varphi(P,Q)$, remove the first step of P, remove $P_{3,4}$ and $P_{6,7}$ and add $\varphi(Q_{2,4})$ and $\varphi(Q_{5,7})$, obtained by shifting $\mathrm{tr}(Q_{2,4})$ and $\mathrm{tr}(Q_{5,7})$ up by one step.

Given a maximal squiggly P-line $P_{i,j}$, the definition of $\widetilde{\mathrm{Dyck}}_n$ implies that $Q_{i-1,j}$ is a sub-Dyck path of Q, from $(i-1,i-1)$ to (j,j). Indeed, on one hand $P_{i,j+1}$ is not a squiggly line, so either $(j,j) = (n,n)$ or $(j+1,j+1) \notin P$, and both situations imply that $(j,j) \in Q$; and on the other hand, $P_{i-1,j}$ is not a squiggly line, so either $(i,i) = (1,1)$ or $(i-1,i-1) \notin P$, and both situations imply that $(i-1,i-1) \in Q$. Let $\mathrm{tr}(Q_{i-1,j})$ be the *truncation* of $Q_{i-1,j}$ obtained by removing the first r-step and last u-step of $Q_{i-1,j}$. Let $\varphi(Q_{i-1,j})$ be the upward shift of $\mathrm{tr}(Q_{i-1,j})$, obtained by shifting each step of $\mathrm{tr}(Q_{i-1,j})$ by one u-step.

For $(P,Q) \in \widetilde{\mathrm{Dyck}}_n$, define $\varphi(P,Q) \in \mathrm{Path}(n,n-1)$ by replacing each maximal squiggly P-line $P_{i,j}$ by $\varphi(Q_{i-1,j})$ and removing the first u step of P. This defines a function:

$$\varphi\colon \widetilde{\mathrm{Dyck}}_n \to \mathrm{Path}(n,n-1).$$

An example is given in Figure 4.

To recover (P,Q) from $\varphi(P,Q)$, it suffices to recover which parts of $\varphi(P,Q)$ come from P. Let $l = \{(x,y) \mid y = x+1\}$ be the upward-shifted diagonal (in red in Figure 4). One checks that a step in $\varphi(P,Q)$ is a part of P if and only if it belongs to a sub-path of the following form:

- the first step of $\varphi(P,Q)$, if this first step is an r-step;
- a sub-path of the form uTr, where T is a sub-path that lies entirely above l.

This shows that φ admits an inverse and concludes. $\square$

Proof of Theorem 3.1. It is clear that the cardinality of $\mathrm{Path}(n,n-1)$ is $\binom{2n-1}{n}$. Combining Lemma 3.3 with Lemma 3.4 concludes. $\square$

4. Background on quantum $\mathfrak{gl}(1|1)$ and its representations

For a comprehensive exposition of the Hopf superalgebra $U_q(\mathfrak{gl}(1|1))$ and its representations, see Sections 2 and 3 of [29]. We stress that our objects are of "super" type and therefore signs appear.

4.1. **Quantum $\mathfrak{gl}(1|1)$.** Fix a basis $\{\varepsilon_1, \varepsilon_2\}$ of the weight lattice $P = \mathbb{Z}^2$ with a pairing given by

$$\langle \varepsilon_i, \varepsilon_j \rangle = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j = 1, \\ -1 & \text{if } i = j = 2. \end{cases}$$

This allows us to define by duality the coweight lattice $P^* = \mathbb{Z}^2$ with basis $\{h_1, h_2\}$. We will work over the field $\mathbb{Q}(q)$, but one can also work oven any field of characteristic zero with an element $q \neq 0$ which is not a root of unity.

Definition 4.1. The *quantum superalgebra* U_q is the associative unital $\mathbb{Q}(q)$-superalgebra with odd generators E, F and even generators $K_1^{\pm 1}, K_2^{\pm 1}$, subject to the relations

$$K_1 K_2 = K_2 K_1, \qquad K_i^{\pm 1} K_i^{\mp 1} = 1, \tag{25}$$
$$K_1 E = q E K_1, \qquad K_2 E = q^{-1} E K_2, \tag{26}$$
$$K_1 F = q^{-1} F K_1, \qquad K_2 F = q F K_2, \tag{27}$$
$$EF + FE = \frac{K - K^{-1}}{q - q^{-1}}, \qquad E^2 = F^2 = 0, \tag{28}$$

where $K = K_1 K_2$.

For $h = n_1 h_1 + n_2 h_2 \in P^*$ we set $K_h = K_1^{n_1} K_2^{n_2}$, so that $K_1 = K_{h_1}$, $K_2 = K_{h_2}$ and $K = K_{h_1 + h_2}$ (note that K is central). Note also that $K_h^{-1} = K_{-h}$. The superalgebra U_q is actually a Hopf superalgebra, with comultiplication Δ, counit ε and antipode S given below.

$$\Delta(K_h) = K_h \otimes K_h, \qquad \Delta(E) = E \otimes K + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \tag{29}$$
$$\epsilon(K_h) = 1, \qquad \epsilon(E) = 0, \qquad \epsilon(F) = 0, \tag{30}$$
$$S(K_h) = K_h^{-1}, \qquad S(E) = -EK^{-1}, \qquad S(F) = -KF. \tag{31}$$

Here, the conventions for the coproduct differ slightly from [29, (2.9)] and are rather in line with [36].

Define a bar involution on U_q by $\overline{E} = E$, $\overline{F} = F$, $\overline{K_h} = K_h^{-1}$ and $\overline{q} = q^{-1}$. Then $\overline{\Delta} := \overline{(\cdot)} \otimes \overline{(\cdot)} \circ \Delta \circ \overline{(\cdot)}$ is also a coproduct with

$$\begin{aligned} \overline{\Delta}(E) &= E \otimes K^{-1} + 1 \otimes E, \\ \overline{\Delta}(F) &= F \otimes 1 + K \otimes F, \\ \overline{\Delta}(K) &= K \otimes K. \end{aligned}$$

Denote $|x|$ the parity of $x \in U_q$ and set $\Delta^{\mathrm{op}}(x) = \sum (-1)^{|x_1| \cdot |x_2|} x_{(2)} \otimes x_{(1)}$ in the Sweedler notation.

4.2. **Representations.** Set $|\varepsilon_1| = 0$ and $|\varepsilon_2| = 1$ and extend linearly to a map $|\bullet| : P \to \mathbb{Z}/2\mathbb{Z}$. We only consider finite dimensional weight representations: these consist of finite dimensional U_q-supermodules $M = \bigoplus_{\lambda \in P} M_\lambda$ with

$$M_\lambda = \{v \in M \colon K_h \cdot v = q^{\langle h, \lambda \rangle} v\}$$

and M_λ in (super-)degree $|\lambda|$

All simple representations of U_q are one- or two-dimensional and are indexed by their highest weight $\lambda \in P$.

- If $\langle h_1{+}h_2, \lambda \rangle = 0$ then the simple representation with highest weight λ is one-dimensional: $\mathbb{Q}(q)_\lambda = \mathbb{Q}(q)v_\lambda$, with $|v_\lambda| = |\lambda|$, and
$$E.v_\lambda = F.v_\lambda = 0, \qquad K_h.v_\lambda = q^{\langle h, \lambda \rangle} v_\lambda.$$
- If $\langle h_1{+}h_2, \lambda \rangle \neq 0$ then the simple representation with highest weight λ is two-dimensional: $L(\lambda) = \mathbb{Q}(q)v_\lambda^0 \oplus \mathbb{Q}(q)v_\lambda^1$ with $|v_\lambda^0| = |\lambda|$, $|v_\lambda^1| = |\lambda| + 1$ and
$$E.v_\lambda^0 = 0, \qquad F.v_\lambda^0 = [\langle h_1 + h_2, \lambda \rangle] v_\lambda^1, \qquad K_h.v_\lambda^0 = q^{\langle h, \lambda \rangle} v_\lambda^0, \tag{32}$$
$$E.v_\lambda^1 = v_\lambda^0, \qquad F.v_\lambda^1 = 0, \qquad K_h.v_\lambda^1 = q^{\langle h, \lambda - \varepsilon_1 + \varepsilon_2 \rangle} v_\lambda^1. \tag{33}$$

where $[k] := \frac{q^k - q^{-k}}{q - q^{-1}}$ is the quantum number.

Example 4.2. The representation $V := L(\varepsilon_1)$ is called the *vector representation.* In this case $\langle h_1 + h_2, \varepsilon_1 \rangle = 1$ and hence also $[\langle h_1 + h_2, \varepsilon_1 \rangle] = 1$. Furthermore $q^{\langle h, \lambda - \varepsilon_1 + \varepsilon_2 \rangle} = q^{\langle h, \varepsilon_2 \rangle}$. To ease notations, we will drop the subscript ε_1 in the vectors $v_{\varepsilon_1}^0$ and $v_{\varepsilon_2}^1$ of the vector representation V.

The tensor product of two-dimensional simple representations follows an easy rule. If $\lambda, \mu \in P$ are such that $\langle h_1 + h_2, \lambda \rangle$, $\langle h_1 + h_2, \mu \rangle$ and $\langle h_1 + h_2, \lambda + \mu \rangle$ are nonzero then

$$L(\lambda) \otimes L(\mu) \cong L(\lambda + \mu) \oplus L(\lambda + \mu - \varepsilon_1 + \varepsilon_2). \tag{34}$$

If $\langle h_1 + h_2, \lambda + \mu \rangle = 0$, then the representation $L(\lambda) \otimes L(\mu)$ is indecomposable.

4.3. **Braiding and Schur–Weyl duality.**

4.3.1. *The quasi-R-matrix and braiding.* The monoidal category of finite dimensional weight representations can be endowed with a braided structure. For this, we introduce the quasi-R-matrix Θ defined by

$$\Theta = 1 \otimes 1 - (q - q^{-1}) E \otimes F.$$

Since our coproduct is not the one of [29], we also have a different quasi-R-matrix. We also define a morphism of superalgebras $\Psi \colon U_q \otimes U_q \to U_q \otimes U_q$ by

$$\Psi(K_h \otimes 1) = K_h \otimes 1, \qquad \Psi(E \otimes 1) = E \otimes K^{-1}, \qquad \Psi(F \otimes 1) = F \otimes K,$$
$$\Psi(1 \otimes K_h) = 1 \otimes K_h, \qquad \Psi(1 \otimes E) = K^{-1} \otimes E, \qquad \Psi(1 \otimes F) = K \otimes F.$$

From here and onward, given an element $x = \sum_i a_i \otimes b_i \in U_q^{\otimes 2}$, we denote by x_{12} the element $\sum_i a_i \otimes b_i \otimes 1$, by x_{13} the element $\sum_i a_i \otimes 1 \otimes b_i$ and by x_{23} the element $\sum_i 1 \otimes a_i \otimes b_i$. We use a similar notation for morphisms. The following are straightforward calculations.

Proposition 4.3. *Let $x \in U_q$. We have*

(1) $\Theta\Delta(x) = \overline{\Delta}(x)\Theta$,
(2) $\overline{\Delta}(x) = \Psi(\Delta^{\mathrm{op}}(x))$,
(3) $\Delta \otimes \mathrm{id}(\Theta) = \Psi_{23}(\Theta_{13})\Theta_{23}$,
(4) $\mathrm{id} \otimes\Delta(\Theta) = \Psi_{12}(\Theta_{13})\Theta_{12}$.

4.3.2. *Braided structure on the category of representations.* The above quasi-R-matrix Θ is used to endow the category of representations of U_q with a braided structure. For M and N two weight modules we define $\Theta_{M,N} \colon M \otimes N \to M \otimes N$ and $f_{M,N} \colon M \otimes N \to M \otimes N$ by

$$\Theta_{M,N}(m \otimes n) = m \otimes n - (-1)^{|m|}(q - q^{-1})E.m \otimes F.n,$$

for $m \in M$ and $n \in N$, and

$$f_{M,N}(m \otimes n) = q^{(\mu,\nu)} m \otimes n$$

if in addition $m \in M_\mu$ and $n \in N_\nu$.

Proposition 4.4. *We have that $f_{M,N} \circ \Theta_{M,N}$ intertwines Δ and Δ^{op}:*

$$\forall x \in U_q, f_{M,N} \circ \Theta_{M,N} \circ \Delta(x) = \Delta^{\mathrm{op}}(x) \circ f_{M,N} \circ \Theta_{M,N}.$$

Proof. This follows from Proposition 4.3.(1) and Proposition 4.3.(2). □

Now set $R_{M,N} = f_{M,N} \circ \Theta_{M,N}$ and $\check{R}_{M,N} = \tau \circ f_{M,N} \circ \Theta_{M,N}$, where τ is the super-twist, which is defined by $\tau(m \otimes n) = (-1)^{|m|\cdot|n|} n \otimes m$.

Theorem 4.5. *The map $\check{R}$ is a braiding in the category of U_q weight representations.*

Proof. The map $\check{R}_{M,N}$ is U_q-linear thanks to Proposition 4.4. The hexagon axioms follow from Proposition 4.3.(3) and Proposition 4.3.(4). □

4.3.3. *The vector representation and a Schur–Weyl duality.* Recall the vector representation $V = L(\varepsilon_1)$. An iterated use of (34) yields

$$V^{\otimes m} \cong \bigoplus_{\ell=0}^{m-1} L((m-\ell)\varepsilon_1 + \ell\varepsilon_2)^{\oplus\binom{m-1}{\ell}}. \tag{35}$$

Thanks to the braiding, the representation $V^{\otimes m}$ is acted upon by the braid group on m strands, and this action factors through the Hecke algebra $H_q(\mathrm{S}_m)$. To be more precise, the map

$$\sigma_i \mapsto q^{-1}\,\mathrm{id}^{\otimes i} \otimes \check{R}_{V,V} \otimes \mathrm{id}^{\otimes(n-i-2))}$$

defines an action of the braid group which factors through the relation $\sigma_i^2 = (1-q^2)\sigma_i + q^2\,\mathrm{id}$.

A Schur–Weyl duality between U_q and the Hecke algebra $H_q(\mathrm{S}_m)$ has been shown independently by Moon [24] and Mitsuhashi [23].

Theorem 4.6. *The algebra U_q and $H_q(\mathrm{S}_m)$ centralize each other in $\mathrm{End}(V^{\otimes m})$; that is, the action of $H_q(\mathrm{S}_m)$ generates $\mathrm{End}_{U_q}(V^{\otimes m})$ and the action of U_q generates $\mathrm{End}_{H_q(\mathrm{S}_m)}(V^{\otimes m})$.*

More precisely, there is a decomposition

$$V^{\otimes m} \cong \bigoplus_{\ell=1}^{m-1} L(m\varepsilon_1 - \ell(\varepsilon_1 - \varepsilon_2)) \otimes \mathcal{S}_{hk(m-\ell,\ell+1)} \tag{36}$$

as $U_q \otimes H_q(\mathrm{S}_m)$-representation, where $\mathcal{S}_{hk(m-\ell,\ell+1)}$ is the Specht module for the hook partition with length $m-\ell$ and height $\ell+1$.

To complete the picture, we exhibit the matrix of $q^{-1}\check{R}_{V,V}$ on the basis

$$\{v^0 \otimes v^0, v^1 \otimes v^0, q^{-1}v^0 \otimes v^1, q^{-1}v^1 \otimes v^1\}.$$

We recover the matrix for the Burau representation of the braid group (with $t = q^{-2}$):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1-t) & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}. \tag{37}$$

Note that the chosen basis is the tensor product of two different bases of V: $\{v^0, v^1\}$ and $\{v^0, q^{-1}v^1\}$.

5. A Schur–Weyl duality with $U_q(\mathfrak{gl}(1|1))^{\leq 0}$

The goal is now to enhance the Schur–Weyl duality between the Hecke algebra and U_q to a Schur–Weyl duality involving the loop Hecke algebra. We denote by $U_q^{\leq 0}$ the subalgebra of U_q generated by K_1, K_2 and F.

5.1. An LB_n-representation via an R-matrix for $U_q^{\leq 0}$.

Inspired by the notion of a twist in a quasi-triangular Hopf algebra, as introduced in [27], we define, for any two U_q weight modules M and N, a map $S_{M,N} \colon M \otimes N \to M \otimes N$ by

$$S_{M,N}(m \otimes n) = q^{\mu_1\nu_2 - \mu_2\nu_1} m \otimes n,$$

where $m \in M_\mu$ and $N \in N_\nu$.

Proposition 5.1. *We have that $S_{M,N}$ intertwines Δ and Δ^{op} on $U_q^{\leq 0}$:*

$$\forall x \in U_q^{\leq 0}, S_{M,N} \circ \Delta(x) = \Delta^{\mathrm{op}}(x) \circ S_{M,N}.$$

Proof. The calculations for $x = K_1$ and $x = K_2$ are trivial. For $x = F$, $m \in M_\mu$ and $n \in N_\nu$, we have:

$$\begin{aligned} S_{M,N}(\Delta(F) \cdot m \otimes n) &= S_{M,N}(Fm \otimes n + (-1)^{|m|} q^{-\mu_1-\mu_2} m \otimes Fn) \\ &= q^{(\mu_1-1)\nu_2 - (\mu_2+1)\nu_1} Fm \otimes n + (-1)^{|m|} q^{\mu_1(\nu_2+1) - \mu_2(\nu_1-1) - \mu_1 - \mu_2} m \otimes Fn \\ &= q^{\mu_1\nu_2 - \mu_2\nu_1}(q^{-\nu_1-\nu_2} Fm \otimes n + (-1)^{|m|} m \otimes Fn) \\ &= \Delta^{\mathrm{op}}(F)(S_{M,N}(m \otimes n)), \end{aligned}$$

the equalities being obtained from the definition of $S_{M,N}$ and of Δ. □

Remark 5.2. The map $S_{M,N}$ does not intertwine Δ and Δ^{op} on U_q. One may check that $S_{M,N} \circ \Delta(E) \neq \Delta^{\mathrm{op}}(E) \circ S_{M,N}$ but $S_{M,N} \circ \Delta^{\mathrm{op}}(E) = \Delta(E) \circ S_{M,N}$.

We now set $\check{S}_{M,N} = \tau \circ S_{M,N}$, where τ is the super-twist, as usual.

Proposition 5.3. *The map $\check{S}$ is a symmetric braiding on the category of $U_q^{\leq 0}$ weight representations.*

Proof. It remains to check the hexagon axioms and that $\check{S}_{N,M} \circ \check{S}_{M,N}$ is the identity. Given three weight modules M, N and L and $m \in M_\mu, n \in N_\nu$ and $\ell \in L_\lambda$, we have

$$\check{S}_{M\otimes N,L}(m \otimes n \otimes \ell) = q^{(\mu_1+\nu_1)\lambda_2-(\mu_2+\nu_2)\lambda_1}(-1)^{(|m|+|n|)|\ell|}\ell \otimes m \otimes n,$$

and

$$\begin{aligned}(\check{S}_{M,L} \otimes \mathrm{id}_N) \circ (\mathrm{id}_M \otimes \check{S}_{N,L})(m \otimes n \otimes \ell) &= q^{\nu_1\lambda_2-\nu_2\lambda_1}(-1)^{|n||\ell|}\check{S}_{M,L}(m \otimes \ell) \otimes n \\ &= q^{\nu_1\lambda_2-\nu_2\lambda_1+\mu_1\lambda_2-\mu_2\lambda_1}(-1)^{|n||\ell|+|m||\ell|}\ell \otimes m \otimes n,\end{aligned}$$

which shows that the hexagon axiom $\check{S}_{M\otimes N,L} = (\check{S}_{M,L} \otimes \mathrm{id}_N) \circ (\mathrm{id}_M \otimes \check{S}_{N,L})$ holds. The proof for the second hexagon axiom is similar. The axiom of symmetry is easy and omitted. □

The goal is now to show some mixed relations satisfied by $\check{R}$ and $\check{S}$.

Proposition 5.4. *Given three $U_q^{\leq 0}$ weight modules M, N and L, we have*

(1) $(\check{S}_{N,L}\otimes\mathrm{id}_M)\circ(\mathrm{id}_N \otimes\check{R}_{M,L})\circ(\check{R}_{M,N}\otimes\mathrm{id}_L) = (\mathrm{id}_L \otimes\check{R}_{M,N})\circ(\check{R}_{M,L}\otimes\mathrm{id}_N)\circ(\mathrm{id}_M \otimes\check{S}_{N,L})$,
(2) $(\check{R}_{N,L}\otimes\mathrm{id}_M)\circ(\mathrm{id}_N \otimes\check{S}_{M,L})\circ(\check{S}_{M,N}\otimes\mathrm{id}_L) = (\mathrm{id}_L \otimes\check{S}_{M,N})\circ(\check{S}_{M,L}\otimes\mathrm{id}_N)\circ(\mathrm{id}_M \otimes\check{R}_{N,L})$.

Proof. We start by noticing that

$$(\mathrm{id}_M \otimes\check{S}_{N,L}) \circ \Theta_{M,N\otimes L} = \Theta_{M,L\otimes N} \circ (\mathrm{id}_M \otimes\check{S}_{N,L}). \tag{38}$$

Indeed, the maps $\Theta_{M,N\otimes L}$ and $\Theta_{M,L\otimes N}$ are induced by the action of $1\otimes\Delta(1)-(q-q^{-1})E\otimes\Delta(F)$, and Proposition 5.1 implies that, for any $m \in M, n \in N$ and $\ell \in L$,

$$\begin{aligned}&(\mathrm{id}_M \otimes S_{N,L}) \circ \Theta_{M,N\otimes L}(m \otimes n \otimes \ell) \\ &\qquad = m \otimes S_{N,L}(n \otimes \ell) - (q-q^{-1})E \otimes \Delta^{\mathrm{op}}(F) \cdot (m \otimes S_{N,L}(n \otimes \ell)).\end{aligned}$$

It remains to apply the super-twist τ_{23} in order to obtain (38).

Now, we have

$$\begin{aligned}(\check{S}_{N,L} \otimes \mathrm{id}_M) \circ \check{R}_{M,N\otimes L} &= \tau_{23} \circ \tau_{12} \circ (\mathrm{id}_M \otimes\check{S}_{N,L}) \circ f_{M,N\otimes L} \circ \Theta_{M,N\otimes L} \\ &= \tau_{23} \circ \tau_{12} \circ f_{M,N\otimes L} \circ (\mathrm{id}_M \otimes\check{S}_{N,L}) \circ \Theta_{M,N\otimes L} \\ &= \tau_{23} \circ \tau_{12} \circ f_{M,L\otimes N} \circ \Theta_{M,L\otimes N} \circ (\mathrm{id}_M \otimes\check{S}_{N,L}) \\ &= \check{R}_{M,N\otimes L} \circ (\mathrm{id}_M \otimes\check{S}_{N,L}),\end{aligned}$$

the second equality is due to the fact that $\mathrm{id}_M \otimes S_{N\otimes L}$ and $f_{M,N\otimes L}$ commutes and the third equality is a consequence of (38). We finally obtain (1) using the hexagon axiom for the braiding $\check{R}$. The proof of (2) is similar and ommited. □

Now consider the assignment

$$\Psi_n : \mathrm{LB}_n \to \mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n}) : \begin{cases} \sigma_i \mapsto \mathrm{id}_V^{\otimes i-1} \otimes \check{R}_{V,V} \otimes \mathrm{id}_V^{\otimes (n-i-1} \\ \rho_i \mapsto \mathrm{id}_V^{\otimes i-1} \otimes \check{S}_{V,V} \otimes \mathrm{id}_V^{\otimes (n-i-1)} \end{cases} \tag{39}$$

Theorem 5.5. *The triple $(V, \check{R}, \check{S})$ is a loop braided vector space, i.e. the map Ψ_n constructed in (39) defines a well-defined representation of LB_n. Moreover, Ψ_n factors through $\mathcal{LH}_n$ for $t = q^{-2}$.*

Proof. That Ψ_n is a well-defined map on LB_n follows from Theorem 4.5, Proposition 5.3 and Proposition 5.4 applied to the vector representation V.

To see that Ψ_n factors through $\mathcal{LH}_n$, we need to check the quadratic relations, which we compute directly on $V \otimes V$. In the basis

$$\{v^0 \otimes v^0, v^1 \otimes v^0 + q^{-1} v^0 \otimes v^1, v^1 \otimes v^0 - q v^0 \otimes v^1, v^1 \otimes v^1\}$$

of $V \otimes V$, which is a basis realizing the decomposition $V \otimes V \simeq L(2\varepsilon_1) \oplus L(\varepsilon_1 - \varepsilon_2)$, we find that the matrices of $q^{-1}\check{R}_{V,V}$ and of $\check{S}_{V,V}$ are respectively

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -q^{-2} & 0 \\ 0 & 0 & 0 & -q^{-2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -q(q-q^{-1}) & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The quadratic relations then follow immediately from a matrix calculation. □

Remark 5.6. The representation $\mathbf{F}_n$ defined in (13) (by [12]) coincides with Ψ_n. Indeed, the basis $\{v^0_{\varepsilon_1} \otimes v^0_{\varepsilon_1}, v^1_{\varepsilon_1} \otimes v^0_{\varepsilon_1}, q^{-1} v^0_{\varepsilon_1} \otimes v^1_{\varepsilon_1}, q^{-1} v^1_{\varepsilon_1} \otimes v^1_{\varepsilon_1}\}$ of $V \otimes V$, the respective matrices of $q^{-1}\check{R}_{V,V}$ and $\check{S}_{V,V}$ are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1-q^{-2}) & q^{-2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

These matrices correspond to the matrices in [12] for $t = q^{-2}$.

5.2. **Schur–Weyl duality for the loop Hecke algebra.** Recall that V denotes the vector representation $L(\epsilon_1)$ of U_q. The aim of the remainder of this section is to obtain the following Schur–Weyl type statement.

Theorem 5.7. *If $q \neq \pm 1$, then the morphism Ψ_n defined in (39) yields a $\mathbb{Q}(q)$-algebra isomorphism from $\mathcal{LH}_n$ to $\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n})$.*

To obtain Theorem 5.7 we need to compute $\dim \mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n})$. To do so we will decompose $\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n})$ in smaller pieces which will be proven in Section 6.1 to be meaningful pieces of the Wedderburn–Mal'cev decomposition of the endomorphism ring.

First we note that the decomposition in (35) is not one of simple $U_q^{\leq 0}$-modules as each $L((m-\ell)\varepsilon_1 + \ell\varepsilon_2)$ has a submodule:

Lemma 5.8. *For any $0 \leq \ell \leq n-1$ the $U_q^{\leq 0}$-modules $L((n-\ell)\varepsilon_1 + \ell\varepsilon_2)$ are indecomposable, but non-semisimple. They have a composition series*

$$\{0\} \subsetneq \mathbb{Q}(q)v_\varepsilon^1 \subsetneq L((n-\ell)\varepsilon_1 + \ell\varepsilon_2).$$

Furthermore, the basis elements v_ε^0 and v_ε^1 have weight $(n-\ell)\varepsilon_1 + \ell\varepsilon_2$, resp. $(n-\ell-1)\varepsilon_1 + (\ell+1)\varepsilon_2$.

Proof. The weights of v_ε^0 and v_ε^1 are recorded in (32) and (33). From these equations we also see that $\mathbb{Q}(q)v_\varepsilon^1$ is indeed a $U_q^{\leq 0}$-submodule. As it is 1-dimensional it is simple. Moreover, a direct computation yields that it is the unique submodule and hence it equals the socle of $L((n-\ell)\epsilon_1 + \ell\epsilon_2)$ which consequently is not semisimple, but indecomposable. □

(AL: It feels more natural to have some statement classifying the simple weight $U_q^{\leq 0}$-modules (we also need to keep the wording module or representation throughout). Since $F^2 = 0$, it is easily seen that the simple weight modules are of 1-dimensional, and therefore labeled by their weights. Once this done, we go through the restriction of the simple U_q-modules, and we give the (obvious) composition factors, since we will use it. Then, we can turn to the restriction of the tensor space.)

For ease of *notation* we denote for the remainder of the paper

$$U_q^{\leq 0} := U_q(\mathfrak{gl}(1|1))^{\leq 0} \text{ and } L(\ell) := L((n-\ell)\varepsilon_1 + \ell\varepsilon_2)). \tag{40}$$

Using (35) we obtain that

$$\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n}) = \bigoplus_{0\leq \ell,k\leq n-1} \mathrm{Hom}_{U_q^{\leq 0}}\left(L(\ell), L(k)\right)^{\oplus \binom{n-1}{\ell}\binom{n-1}{k}}. \tag{41}$$

We will now determine the dimension of each summand in (41). Firstly, Lemma 5.8 tells that $L((n-\ell)\varepsilon_1+\ell\varepsilon_2)$ and $L((n-k)\varepsilon_1+k\varepsilon_2)$ have no common weight if $|k-\ell| > 1$. More precisely, then their composition series has no common factor and hence the Hom-space is zero. Thus it remains to consider the case that $|k-\ell| \leq 1$.

Proposition 5.9. *For any $n \in \mathbb{N}_{\geq 1}$ one has that*

$$\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n}) = \bigoplus_{\ell=0}^{n-1} \mathrm{End}_{U_q^{\leq 0}}(L(\ell))^{\oplus \binom{n-1}{\ell}^2} \oplus \bigoplus_{\ell=1}^{n-1} \mathrm{Hom}_{U_q^{\leq 0}}\left(L(\ell), L(\ell-1)\right)^{\oplus \binom{n-1}{\ell}\binom{n-1}{\ell-1}}$$

wtih every summand a 1-dimensional $\mathbb{Q}(q)$-vector space. Furthermore:

$$\dim \mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n}) = \binom{2n-1}{n}.$$

Proof. Consider a summand $\mathrm{Hom}_{U_q^{\leq 0}}\left(L(\ell), L(k)\right)$ from (41). First suppose that $k = \ell$, i.e. consider $\mathrm{End}_{U_q^{\leq 0}}(L(\ell))$. As they are endomorphisms of multiplicity-free finite length modules we obtain that

$$\mathrm{End}_{U_q^{\leq 0}}(L(\ell)) = \mathbb{Q}(q)\mathrm{Id}. \tag{42}$$

Next, let $k = \ell + 1$. By Lemma 5.8, $L(\ell)$ has basis vectors v_ℓ^0 (weight $(n-\ell)\varepsilon_1 + \ell\varepsilon_2$) and v_ℓ^1 (weight $(n-\ell-1)\varepsilon_1 + (\ell+1)\varepsilon_2$). Similarly $L(\ell+1)$ has basis vectors $v_{\ell+1}^0$ (weight

$(n-\ell-1)\varepsilon_1 + (\ell+1)\varepsilon_2)$ and $v^1_{\ell+1}$ (weight $(n-\ell-2)\varepsilon_1 + (\ell+2)\varepsilon_2)$. Therefore, a weight argument shows that a non-trivial $\varphi \in \mathrm{Hom}_{U_q^{\leq 0}}(L(\ell), L(\ell+1))$ satisfy $\varphi(v^0_\ell) = 0$ and should send v^1_ℓ onto a multiple of $v^0_{\ell+1}$. However recall that $\mathbb{Q}(q)v^1_\ell = Rad(L(\ell))$. Since radicals are invariant, one should have that $\varphi(v^1_\ell) \in Rad(L(\ell+1)) = \mathbb{Q}(q)v^1_{\ell+1}$. In conclusion, $\varphi = 0$.

Finally let $k = \ell - 1$. Now $L(\ell-1)$ basis vectors $v^0_{\ell-1}$ (weight $(n-\ell+1)\varepsilon_1 + (\ell-1)\varepsilon_2$) and $v^1_{\ell-1}$ (weight $(n-\ell)\varepsilon_1 + \ell\varepsilon_2$). Again by a weight argument a non-trivial $\varphi \in \mathrm{Hom}_{U_q^{\leq 0}}(L(\ell), L(\ell-1))$ must satisfy $\varphi(v^1_\ell) = 0$ and $\varphi(v^0_\ell) \in \mathbb{Q}(q)v^1_{\ell-1}$. This time the latter is no contradiction with preservation of the radical. In conclusion

$$\mathrm{Hom}_{U_q^{\leq 0}}(L(\ell), L(\ell-1)) = \left\{ \varphi : \left\{ \begin{array}{l} v^0_\ell \mapsto z.v^1_{\ell-1} \\ v^1_\ell \mapsto 0 \end{array} \right. \mid z \in \mathbb{Q}(q) \right\} \cong \mathbb{Q}(q) \tag{43}$$

which is 1-dimensional, as desired.

It remains to prove the moreover part. This readily follows by taking dimensions in the vector space decomposition just obtained:

$$\dim \mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n}) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell}^2 + \sum_{\ell=1}^{n-1} \binom{n-1}{\ell} . \binom{n-1}{\ell-1} = \binom{2(n-1)}{n-1} + \binom{2(n-1)}{n-2}$$

where in the second equation we have used well-known binomial coefficient identities. It is also classical that the last expression equals $\binom{2n-1}{n}$, finishing the proof. $\square$

We have now the necessary tools to conclude.

Proof of Theorem 5.7. By Theorem 5.5 the representation Ψ_n coincides with $\mathbf{F}_n$ defined in (13). It was proven in [12, Theorem 5.8] that $\dim \mathrm{im}(\mathbf{F}_n) = \binom{2n-1}{n}$ which equals $\dim \mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n})$ by Proposition 5.9. Hence $\Psi_n = \mathbf{F}_n$ is surjective. Now if $q \neq \pm 1$, then $\mathcal{LH}_n$ is isomorphic to $\widetilde{\mathcal{LH}_n}$ by Theorem 7.1. Solely using that the $\widetilde{\mathcal{LH}_n}$-reduced words $\mathrm{Red}(\widetilde{\mathcal{LH}_n})$ generate $\widetilde{\mathcal{LH}_n}$ (cf. Lemma 2.4), we obtain that $\dim \mathcal{LH}_n \leq |\mathrm{Red}(\widetilde{\mathcal{LH}_n})|$ with the upper bound equal to $\binom{2n-1}{n}$ by Theorem 3.1. Thus comparing dimensions with the co-domain we see that Ψ_n must be an isomorphism. $\square$

Remark 5.10. Note that the proof Theorem 5.7 yields an alternative way to obtain that the $\widetilde{\mathcal{LH}_n}$-reduced words are linearly independent.

6. A ring and representation theoretical perspective on loop Hecke algebra

In this section we make use of Theorem 5.7 to study the loop Hecke algebra in more detail through the isomorphic algebra $\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n})$.

6.1. **Description of the semisimple part and Jacobson radical.** If A is a finite dimensional algebra over a perfect field $\Bbbk$, then the theorem of Wedderburn–Mal'cev yields that there exists a maximal semisimple subalgebra B_{ss} in A such that

$$A \cong B_{ss} \oplus Jac(A) \tag{44}$$

as a $\Bbbk$-vector space. In (44) $Jac(A)$ denotes the Jacobson radical of A and B_{ss} is unique up to an algebra automorphism of A. The aim of this section is to describe the constituents of (44) for the loop Hecke algebra. More precisely, we will use Theorem 5.7 to instead describe the Wedderburn–Mal'cev decomposition of the isomorphic algebra $\operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n})$.

6.1.1. *The Wedderburn–Mal'cev decomposition and application.* To start, recall the notations $U_q^{\leq 0}$ and $L(\ell)$ from (40) and consider the following constituents of the decomposition obtained in Proposition 5.9:

$$N_\ell := \operatorname{End}_{U_q^{\leq 0}}(L(\ell))^{\oplus\binom{n-1}{\ell}^2} \text{ and } R_{\ell'} := \operatorname{Hom}_{U_q^{\leq 0}}(L(\ell'), L(\ell'-1))^{\oplus\binom{n-1}{\ell'}\binom{n-1}{\ell'-1}}.$$

Theorem 6.1. *For any $n \in \mathbb{N}_{\geq 1}$ and with notations as above, we have that:*

(1) $\bigoplus_{\ell=0}^{n-1} N_\ell$ is a maximal semisimple subalgebra with N_ℓ a simple factor[10]*,*
(2) $\bigoplus_{\ell=0}^{n-1} N_\ell = \operatorname{End}_{U_q}(V^{\otimes n}) \cong$ Super Temperley–Lieb algebra,
(3) $J\big(\operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n})\big) = \bigoplus_{\ell'=1}^{n-1} R_{\ell'}$ and has square-zero[11]

Moreover, for $0 \leq i, j \leq n-1$, denoting by e_i the identity of N_i, we have:

$$e_j \operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n}) e_i = \begin{cases} N_i & \textit{if } j = i, \\ R_i & \textit{if } j = i-1, \\ 0 & \textit{else.} \end{cases}$$

Before proving Theorem 6.1 we discuss some consequences.

Remark 6.2. A consequence of Theorem 6.1 and Theorem 4.6 is that the Hecke algebra $H_n(\mathrm{S}_n)$ surjects onto $\operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n})/J\big(\operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n})\big)$. Now recalling that the Jacobson radical is preserved under morphisms and using Theorem 5.7 we obtain that

$$H_q(\mathrm{S}_n) \twoheadrightarrow \mathcal{LH}_n / J(\mathcal{LH}_n)$$

when $t \neq \pm 1$. It would be interesting to have an explicit description of a basis for the pieces $\Psi^{-1}(N_i)$ and $\Psi^{-1}(R_i)$, i.e. to describe for every $\widetilde{\mathcal{LH}}_n$-reduced word where its image under Ψ lives. Note that it follows from Theorem 4.5 that

$$\Psi(\sigma_\ell) \in N_\ell$$

for every $0 \leq \ell \leq n-1$.

The elements $\{e_0, \ldots, e_{n-1}\}$ form a set of orthogonal idempotents that add up to the identity: $\mathrm{id}_{V^{\otimes n}} = e_0 + \cdots + e_{n-1}$. Thanks to this one can consider the associated Peirce decomposition

$$\operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n}) = \bigoplus_{0 \leq i,j \leq n-1} e_j \operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n}) e_i$$

which allows one to consider $\operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n})$ as a $n \times n$ matrix algebra with $e_j \operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n}) e_i$ as entry (i, j). Associated with this one can consider the $n \times n$-matrix $M = (M_{i,j})_{0 \leq i,j \leq n-1}$ with

$$M_{i,j} := \dim e_j \operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n}) e_i.$$

[10]That N_ℓ is a simple factor means that N_ℓ is a simple ring and a two-sided ideal of $\bigoplus_{\ell=0}^{n-1} N_\ell$.

[11]This means that $J\big(\operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n})\big)^2 = 0$, i.e. the product of any two elements is zero.

In [12] the matrix M is referred to as encoding the "structure of the algebra". In [12, Section 6 & Conjecture 6.4] a conjecture was formulated for the form of M. As a corollary of the above we now confirm their conjecture.

Corollary 6.3. *Denote, for $0 \leq \ell \leq n-1$, by $w_\ell := \binom{n-1}{\ell}$ the ℓ^{th} entry of the n^{th} row of Pascal's triangle. With notations as above, we have that*

$$M = \begin{pmatrix} w_0^2 & 0 & 0 & \cdots & 0 & 0 \\ w_0 w_1 & w_1^2 & 0 & \cdots & 0 & 0 \\ 0 & w_1 w_2 & w_2^2 & \ddots & \vdots & 0 \\ 0 & 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & w_{n-2}^2 & 0 \\ 0 & \cdots & & 0 & w_{n-2}w_{n-1} & w_{n-1}^2 \end{pmatrix} = D. \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & \ddots & \ddots \end{pmatrix} .D$$

where $D := diag(w_0, \cdots, w_{n-1})$.

The second equality in Corollary 6.3 was added as it is in this form that Damiani–Martin–Rowell formulated their conjecture, see matrix $\mathcal{M}_n^p$ in [12, Section 6].

Proof of Corollary 6.3. In Theorem 6.1 the form of the corners $e_j \operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n})e_i$ has been described. Now recall that by Proposition 5.9

$$\dim \operatorname{End}_{U_q^{\leq 0}}(L(\ell)) = \dim \operatorname{Hom}_{U_q^{\leq 0}}(L(\ell'), L(\ell'-1)) = 1$$

for all $0 \leq \ell \leq n-1$ and $1 \leq \ell' \leq n-1$. Hence $\dim N_\ell = \binom{n-1}{\ell}^2$ and $\dim R_{\ell'} = \binom{n-1}{\ell'}\binom{n-1}{\ell'-1}$, which proves the first equality. The second equality is a direct matrix multiplication. $\square$

Remark 6.4. In [12, Conjecture 6.4] the form of the matrix M was only implicitly formulated for the loop Hecke algebra $\mathcal{LH}_n$ itself. Instead, they conjectured that the structure of certain quotients of $\mathcal{LH}_n$ was given by some truncations of the matrix M. In Section 7.2, we will prove that, unexpectedly, these quotients are trivial. Thus, the conjectured form is valid solely when no quotient is applied.

6.1.2. *Towards the proof of Theorem 6.1.* To start, recall (35) saying that

$$V^{\otimes n} \cong \bigoplus_{\ell=0}^{n-1} L((n-\ell)\varepsilon_1 + \ell\varepsilon_2)^{\oplus \binom{n-1}{\ell}}.$$

For the remaining of the section we order the copies of $V^{\otimes n}$ from 1 till $\binom{n-1}{\ell}$. Accordingly denote

$$L_{\ell,k} := k^{th} \text{ copy of } L((n-\ell)\varepsilon_1 + \ell\varepsilon_2) \text{ in } V^{\otimes n}$$

with $1 \leq k \leq \binom{n-1}{\ell}$. Note that Proposition 5.9 now in fact tells that

$$\operatorname{End}_{U_q^{\leq 0}}(V^{\otimes n}) = \bigoplus_{\ell=0}^{n-1} \bigoplus_{1 \leq k,k' \leq \binom{n-1}{\ell}} \operatorname{Hom}_{U_q^{\leq 0}}(L_{\ell,k}, L_{\ell,k'}) \oplus \bigoplus_{\ell'=1}^{n-1} \bigoplus_{r=1}^{\binom{n-1}{\ell'}} \bigoplus_{r'=1}^{\binom{n-1}{\ell'-1}} \operatorname{Hom}_{U_q^{\leq 0}}(L_{\ell',r}, L_{\ell'-1,r'}) \tag{45}$$

Now for $\varphi \in \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell,k}, L_{\ell,k'})$ and $\psi \in \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell',r}, L_{\ell'-1,r'})$ we directly see from (45) that

$$\varphi \circ \psi \neq 0 \Leftrightarrow \ell = \ell' - 1 \text{ and } k = r'. \tag{46}$$

In that case $\varphi \circ \psi \in \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell',r}, L_{\ell'-1,k'})$. Similarly,

$$\psi \circ \varphi \neq 0 \Leftrightarrow \ell' = \ell \text{ and } r = k' \tag{47}$$

in which case $\psi \circ \varphi \in \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell,k}, L_{\ell-1,r'})$.

Proof of Theorem 6.1. In terms of the notations above we have by definition that

$$N_\ell = \bigoplus_{1 \leq k,k' \leq \binom{n-1}{\ell}} \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell,k}, L_{\ell,k'}) \text{ and } R_{\ell'} = \bigoplus_{r=1}^{\binom{n-1}{\ell'}} \bigoplus_{r'=1}^{\binom{n-1}{\ell'-1}} \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell',r}, L_{\ell'-1,r'}).$$

From (46) and (47) it directly follows that $\bigoplus_{\ell'=1}^{n-1} R_{\ell'}$ is a two-sided ideal of $\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n})$ whose square is zero (i.e. the product of any two elements is zero). Those equations also imply that $N_{\ell_1}.N_{\ell_2} = 0$ for $0 \leq \ell_1 \neq \ell_2 \leq n-1$. Hence the vector space decomposition $\bigoplus_{\ell=0}^{n-1} N_\ell$ is also one of rings and in particular N_ℓ is a two-sided ideal of $\bigoplus_{\ell=0}^{n-1} N_\ell$. Since all spaces $\mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell,k}, L_{\ell,k'})$ are 1-dimensional, by Proposition 5.9, (46) and (47) also imply that the two-sided ideal (inside $\bigoplus_{\ell=0}^{n-1} N_\ell$) generated by any $0 \neq \varphi \in \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell,k}, L_{\ell,k'})$ is the full of N_ℓ. From this we infer that N_ℓ is a simple ring, finishing the proof of part (1) of the statement, except the maximality. The latter will follow if we prove that the vector space complement $\bigoplus_{\ell'=1}^{n-1} R_{\ell'}$, see (45), equals the Jacobson radical.

Next note that since $\bigoplus_{\ell'=1}^{n-1} R_{\ell'}$ is a nil-ideal, being of square-zero, it is contained in the Jacobson radical. Recall that for Artinian algebras the Jacobson radical is also characterized as the smallest (two-sided) ideal such that the corresponding quotient is semisimple. Now as $\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n}) / \bigoplus_{\ell'=1}^{n-1} R_{\ell'} \cong \bigoplus_{\ell=0}^{n-1} N_\ell$ is semisimple, by the statement obtained earlier, we obtain the other inclusion and finishing both statements (3) and (1).

Concerning the moreover part, note that

$$e_i = \sum_{1 \leq k,k' \leq \binom{n-1}{\ell}} \mathrm{id}_{k,k'}$$

where $\mathrm{id}_{k,k'}$ is the canonical identification of $L_{\ell,k}$ and $L_{\ell,k'}$. With this description and using (46) and (47) the moreover part follows directly.

Finally, for statement (2) note that (42) shows that $\mathrm{End}_{U_q^{\leq 0}}(L(\ell)) = \mathrm{End}_{U_q}(L(\ell))$. Since the modules $L(\ell)$ are simple U_q-modules, the latter and (35) implies that

$$\bigoplus_{\ell=0}^{n-1} N_\ell = \bigoplus_{\ell=0}^{n-1} \mathrm{End}_{U_q}(L(\ell)^{\oplus \binom{n-1}{\ell}^2}) = \mathrm{End}_{U_q}(V^{\otimes n}).$$

Now it was proven in [28, Section 4] that $\mathrm{End}_{U_q}(V^{\otimes n})$ is isomorphic to the super Temperley–Lieb algebra. $\square$

6.2. **A description of the Ext-quiver and Cartan matrix.** Consider the primitive central idempotents $\{e_0, \ldots, e_{n-1}\}$ described in Theorem 6.1. As the idempotents are orthogonal and $1 = \sum_{i=0}^{n-1} e_i$ one has the associated decomposition in blocks:

$$\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n}) = \bigoplus_{i=0}^{n-1} \mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n}).e_i$$

A block $\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n}).e_i$ can be further decomposed into

$$\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n}).e_i = P_i^{\oplus m_i}$$

where $\{P_0, \ldots, P_{n-1}\}$ are representatives of the isomorphism classses of indecomposable projective (left) modules. Their head $P_i/J(P_i)$ is simple which we denote by S_i. Below, in Proposition 6.6, we will describe these modules explicitly and also the multiplicities m_i.

For a finite dimensional algebra A, the *Cartan matrix* $C(A)$ is an $n \times n$-matrix whose i^{th} row encode the multiplicity of each simple module in the composition series of P_i:

$$C(A)_{i,j} := \dim \mathrm{Hom}_A(P_j, P_i).$$

The Ext-quiver Q_A encode a complementary piece of information: its vertices are given by the simple modules S_i and

$$\#S_i \to S_j := \dim \mathrm{Ext}^1_A(S_i, S_j).$$

In [12, Theorem 5.7 & Corollary 6.1] both were described for the algebra $SP_n := \mathbf{F}_n(k[\mathrm{LB}_n])$ with $\mathbf{F}_n$ defined in (13). When $t \neq \pm 1$ it follows from Theorem 5.5 and Theorem 5.7 that $SP_n \cong \mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n})$ and hence their description also hold for the latter. We now give a direct proof, for any t, for the Cartan matrix and Ext-quiver of $\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n})$.

Theorem 6.5. *Let $A := \mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n})$ for $n \in \mathbb{N}_{\geq 1}$. Then the Cartan matrix is*

$$C(A) = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \ddots \end{pmatrix}$$

and the Ext-quiver Q_A with relations is the A_n-quiver with the composition of two arrows zero:

$$\begin{tikzcd} n-1 \arrow[r] \arrow[rr, dash, dashed, bend left] & n-2 \arrow[r] & n-3 & \cdots & 2 \arrow[r] \arrow[rr, dash, dashed, bend left] & 1 \arrow[r] & 0 \end{tikzcd}$$

The proof of Theorem 6.5 will quickly follow once we constructed explicitly the simple and indecomposable projective modules. To do so, we use the notations from Section 6.1.2 and the statements obtained there.

Define for $1 \leq \ell \leq n-1$ and $1 \leq k \leq \binom{n-1}{\ell}$:

$$S_{\ell,k} := \bigoplus_{r=1}^{\binom{n-1}{\ell-1}} \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell,k}, L_{\ell-1,r}) \text{ and } P_{\ell,k} := S_{\ell,k} \oplus \bigoplus_{k'=1}^{\binom{n-1}{\ell}} \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell,k}, L_{\ell,k'}) \tag{48}$$

If $\ell = 0$, then $\binom{n-1}{\ell} = 1$ and we define $P_{0,1} := \mathrm{Hom}_{U_q^{\leq 0}}(L_{0,1}, L_{0,1})$ and $S_{0,1} = \{0\}$. Note that, in terms of Theorem 6.1, $S_{\ell,k}$ correspond to all endomorphisms in the radical which are zero outside a fixed $L_{\ell,k}$, and $P_{\ell,k}$ to all morphisms in $\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n})$ outside a fixed $L_{\ell,k}$.

Proposition 6.6. *With notations as above we have:*

(1) $P_{\ell,k}$ is an indecomposable projective module and it is cyclic,
(2) $\{0\} \subseteq S_{\ell,k} \subsetneq P_{\ell,k}$ is a composition series,
(3) $Jac(P_{\ell,k}) = Soc(P_{\ell,k}) = S_{\ell,k}$,
(4) $P_{\ell_1,k_1} \cong P_{\ell_2,k_2}$ if and only if $\ell_1 = \ell_2$,
(5) $\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n}).e_\ell \cong P_{\ell,1}^{\oplus m_\ell}$ with $m_\ell = \binom{n-1}{\ell}$.

Proof. Using (46) and (47) it is easily seen that $S_{\ell,k}$ and $P_{\ell,k}$ are left modules. Furthermore by construction, see (45),

$$\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n}) = \bigoplus_{\ell=1}^{n-1} \bigoplus_{k=1}^{\binom{n-1}{\ell}} P_{\ell,k} \tag{49}$$

Thus the complement of $P_{\ell,k}$ in $\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n})$ is also a left-module and hence $P_{\ell,k}$ is projective. Next, again using (46) and (47) is readily verified that $S_{\ell,k}$ is simple and the only submodule of $P_{\ell,k}$. In particular $P_{\ell,k}$ is indecomposable. Furthermore via analogue computations one sees that $P_{\ell,k}/S_{\ell,k}$ is simple and that $P_{\ell,k}$ is generated as left module by any $0 \neq \varphi \in \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell,k}, L_{\ell,k'})$ (for any k'), finishing the proof of statement (1) and (2). Statement (3) follows from the second. Next, assertion (4) holds by definition and the explicit description of the hom-spaces obtained in (43) and (42). Finally, statement (5) follows from the fourth and that by construction $\mathrm{End}_{U_q^{\leq 0}}(V^{\otimes n})e_\ell = \bigoplus_{k=1}^{\binom{n-1}{\ell}} P_{\ell,k}$. □

We now have the necessary tools to prove the main result of this section.

Proof of Theorem 6.5. As representatives for the indecomposable projective modules we take the modules $P_{\ell,1}$ which we denote by P_ℓ for ease of notation.

Consider $0 \leq i, j \leq n-1$. We want to compute $\mathrm{Hom}_{U_q^{\leq 0}}(P_i, P_j)$. Since the modules P_ℓ are cyclic generated by any $0 \neq \varphi_\ell \in \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell,1}, L_{\ell,k'})$, it is enough to determine the image of φ_i. From (46), (47) and the composition series in Proposition 6.6 it now follows that $\mathrm{Hom}_{U_q^{\leq 0}}(P_i, P_j) = 0$ if $i \neq j$ or $j-1$. It also implies that if $i = j$, then $\mathrm{End}(P_i)$ consists of scalar endomorphisms. Finally if $i = j-1$, then any morphism is of the form $\psi^* = (-) \circ \psi$ with $\psi \in \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell,1}, L_{\ell-1,1})$. Note that

$$\psi^*(Soc(P_{j-1})) = 0 \text{ and } \psi^*(P_{j-1}) = Jac(P_i). \tag{50}$$

Altogether we obtain the stated form of the Cartan matrix.

For the Ext-quiver we need to compute the values $\dim \mathrm{Ext}^1(S_{i,1}, S_{j,1})$. Now recall that

$$\dim \mathrm{Ext}^1(S_{i,1}, S_{j,1}) = \dim \mathrm{Hom}_{U_q^{\leq 0}}(P_j, Jac(P_i)/Jac(P_i)^2).$$

From Proposition 6.6 we know that $Jac(P_i) = S_{i,1}$ and consequently $Jac(P_i)^2 = \{0\}$. In particular we need to count morphisms from P_j to P_i. From the computations earlier, mind the switch in the role of i and j, we know that this forces $j = i$ or $i-1$. If $j = i$ the endomorphisms are scalar multiples of the identity and hence the image is not in $Jac(P_i)$. If $j = i-1$, then (50) yields that $\dim \mathrm{Hom}_{U_q^{\leq 0}}(P_j, Jac(P_i)) = 1$, finishing the proof. □

Remark 6.7. One could also have worked with right modules. In that case one has the analogue of Proposition 6.6 but for the following right modules:

$$S^{(r)}_{\ell-1,k'} := \bigoplus_{r=1}^{\binom{n-1}{\ell}} \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell,r}, L_{\ell-1,k'}) \text{ and } P^{(r)}_{\ell-1,k'} := S^{(r)}_{\ell-1,k'} \oplus \bigoplus_{k=1}^{\binom{n-1}{\ell-1}} \mathrm{Hom}_{U_q^{\leq 0}}(L_{\ell-1,k}, L_{\ell-1,k'}) \tag{51}$$

with $0 \leq \ell - 1 \leq n-2$ and $1 \leq k' \leq \binom{n-1}{\ell-1}$. For $\ell - 1 = n-1$ one defines $P_{n-1,1} := \mathrm{Hom}_{U_q^{\leq 0}}(L_{n-1,1}, L_{n-1,1})$.

7. Equivalence between the presentations and consequences

The main aim of this section is to prove the following crucial result.

Theorem 7.1. *The loop Hecke algebra $\mathcal{LH}_n$ (Definition 1.2) and the algebra $\widetilde{\mathcal{LH}}_n$ (Definition 1.4) are isomorphic under the following change of coefficients:*

(1) when we localize at $t, t\pm 1$:

$$\mathcal{LH}_n \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t^{\pm 1}]\left[\frac{1}{t\pm 1}\right] \cong \widetilde{\mathcal{LH}}_n \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]\left[\frac{1}{t\pm 1}\right],$$

(2) when we specialize at $t = 0$:

$$\mathcal{LH}_n \otimes_{\mathbb{Z}[t]} \mathbb{Z} \cong \widetilde{\mathcal{LH}}_n,$$

where $\mathbb{Z}$ is viewed as a $\mathbb{Z}[t]$-algebra with t acting as 0.

In both cases, the isomorphism is explicitly given by the following mutually inverse maps:

$$\begin{cases} D_i & \mapsto (\sigma_i - \rho_i)/(1-t) \\ U_i & \mapsto (\sigma_i - t\rho_i)/(1-t) \end{cases} \quad \leftrightarrows \quad \begin{cases} \sigma_i & \mapsto U_i - tD_i \\ \rho_i & \mapsto U_i - D_i \end{cases}.$$

Corollary 7.2. *Let $\Bbbk$ be a field and $\tilde{t} \in \Bbbk$ with $\tilde{t} \neq \pm 1$. We view $\Bbbk$ as a $\mathbb{Z}[t]$-algebra via the map $t \mapsto \tilde{t}$. Then the algebras $\mathcal{LH}_n \otimes_{\mathbb{Z}[t]} \Bbbk$ and $\widetilde{\mathcal{LH}}_n \otimes_{\mathbb{Z}} \Bbbk$ are isomorphic.*

Combining with Theorem 2.1 and Theorem 3.1, we get:

Corollary 7.3. *Under the change of coefficients (1) and (2) as in Theorem 7.1, the loop Hecke algebra $\mathcal{LH}_n$ is free of rank $\frac{1}{2}\binom{2n}{n}$.*

In what follows, whenever we work over a field $\Bbbk$ as in the corollary above, we write $\tilde{t}$ instead of t.

As an application we describe in Section 7.2 the quotient of $\mathcal{LH}_n$ by the two-sided ideal generated by

$$\chi^{(j+1)} := (\sigma_1 - \rho_1)\cdots(\sigma_j - \rho_j)$$

for $1 \leq j \leq n$. Concretely we obtain that if $t \neq \pm 1$, then $(\mathcal{LH}_n \otimes_{\mathbb{Z}[t]} \Bbbk)/(\chi^{(j)}) \cong \Bbbk$ which disproves [12, Conjecture 6.4]. However as a by-product this yields an interesting counit, i.e. structure of augmented algebra.

Finally, the presentation of the loop Hecke algebra in the generators $\{\rho_i, D_i\}$ has quite some similarities with the Hecke-Hopf algebra $\mathbf{H}(\mathrm{S}_n)$ introduced by Berenstein–Kazhdan [2]. In Section 7.3 we show that $\mathbf{H}(\mathrm{S}_n)$ canonically surjects as an augmented $\mathbb{Z}$-algebra onto $\widetilde{\mathcal{LH}}_n$, but the kernel is not a Hopf ideal. In fact, we check that for small n the loop Hecke algebra is *not* a Hopf algebra, raising the question of whether there exists a variant of the Hecke-Hopf algebra for the loop Hecke algebra.

7.1. **Proof that both presentations are equivalent.** In this section we prove Theorem 7.1. It is readily verified that the maps in Theorem 7.1 are each other's inverses. Hence, it remains to prove that the maps are well-defined.

Now consider the application

$$\phi\colon \mathcal{LH}_n|_{t\neq\pm1} \to \widetilde{\mathcal{LH}}_n|_{t\neq\pm1} : \begin{cases} \sigma_i & \mapsto U_i - tD_i \\ \rho_i & \mapsto U_i - D_i \end{cases}$$

defined in Theorem 7.1. We must show that the image under ϕ of the defining relations in Definition 1.2 imply the defining relations in Definition 1.4, and vice-versa. It is easy to check that the image of the distant-label quadratic relations (4) imply the distant-label quadratic relations (11), and vice-versa.

As in Section 2.1.1, we abuse notation and write

$$U \coloneqq U_i \text{ and } U_+ \coloneqq U_{i+1},$$

and similarly for D, ρ and σ.

Remark 7.4. The relations $D_+DD_+ = D_+D$ and $UU_+U = U_+U$ in the presentation of $\mathcal{LH}_n$ are consequences of the same-label relations and the relations $UD_+ = 0$ and $U_+D = DU_+$. One can see that by simplifying $D_+U_+DD_+$ (resp. UU_+DU) in two different ways.

7.1.1. *A first look at the relations.* The image under ϕ of the same-label relations (5) and (6) in the σ's and ρ's is:

$$(U - tD - 1)(U - tD + t) = 0, \qquad (U - D)^2 - 1 = 0,$$
$$(U - D - 1)(U - tD + t) = 0 \qquad (U - tD - 1)(U - D + 1) = 0.$$

When $t - 1$ is invertible, these relations are equivalent to the same-label relations in the U's and D's (7). Indeed, if we write q_1, q_2, q_3 and q_4 the left-hand side of these relations, we have that:

$$\frac{1}{(t-1)^2}(q_2 - q_4 - q_3 + q_1) = D^2 - D,$$
$$\frac{1}{t-1}(q_3 - q_1) = DU - tD^2 + tD \quad \text{and} \quad \frac{1}{t-1}(q_2 - q_3) = UD - D^2 - U + 1,$$

which concludes using q_2.

On the other hand, the braid relations in the ρ's and σ's (2) and (3) give:

$$(U - aD)(U_+ - bD_+)(U - bD) = (U_+ - bD_+)(U - bD)(U_+ - aD_+),$$

where $(a,b) = (1,1)$, $(1,t)$, $(t,1)$ or (t,t). Expending gives the family of relations:

$$\begin{aligned} r_{(a,b)} \coloneqq\ & UU_+U - U_+UU_+ \\ & - a(DU_+U - U_+UD_+) \\ & - b(UD_+U + UU_+D - U_+DU_+ - D_+UU_+) \\ & + ab(DD_+U + DU_+D - U_+DD_+ - D_+UD_+) \\ & + b^2(UD_+D - D_+DU_+) \\ & - ab^2(DD_+D - D_+DD_+) \end{aligned}$$

Here we say "the relation $r_{(a,b)}$" to refer to the relation $r_{(a,b)} = 0$. One checks that the defining relations of $\widetilde{\mathcal{LH}}_n$ imply the above (family of) relation(s). It remains to show that this (family of) relation(s), together with the same-label relations, implies the remaining defining relations of $\widetilde{\mathcal{LH}}_n$.

We have two cases: (1) the case when t is invertible (Section 7.1.2) and (2) the case $t = 0$ (Section 7.1.3).

7.1.2. *Case (1).*

Step 1. We compute the following relation, simplifying with the same-label relations:

$$\begin{aligned} s_1 \coloneqq\ & \frac{1}{(t-1)^2}\Big[&& (t^3 r_{(1,1)} - t^2 r_{(1,t)} - t\, r_{(t,1)} + r_{(t,t)})\, U \\ & - && (t^3 r_{(1,1)} - t^2 r_{(1,t)} - t^2 r_{(t,1)} + t r_{(t,t)})\, D \\ & + && U_+\, (t^3 r_{(1,1)} - t\, r_{(1,t)} - t^2 r_{(t,1)} + r_{(t,t)}) \\ & - && D_+\, (t^3 r_{(1,1)} - t\, r_{(1,t)} - t^3 r_{(t,1)} + t r_{(t,t)}) \\ & - && (t-1)\, (t\, r_{(1,t)} - r_{(t,t)}) \Big] \\ =\ & tUD_+U - tDU_+U - t^2D_+UD_+ + t^2D_+DU_+. \end{aligned}$$

This gives a new relation between words of length three.

Recall that by assumption of the case at hand, t is invertible. We compute the following, again using the same-label relations to simplify:

$$s_2 \coloneqq \frac{-1}{t^2(t-1)}(D_+s_1 + s_1U - s_1) = D_+UD_+ - D_+DU_+.$$

This gives a relation between two words of length three.

Step 2. We derive further relations from s_1 and s_2. Multiplying s_2 on the right with U_+ gives

$$D_+UD_+ = D_+DU_+ = 0.$$

Combining with s_1 gives $UD_+U = DU_+U$, using the assumption t invertible. Multiplying this relation on the left by D gives

$$UD_+U = DU_+U = 0.$$

In turn, we can derive further relations from the relation above by multiplying on the left and on right in such a way that the pattern UD (or U_+D_+) appears, and simplifying with the relevant same-label relation. This gives:

$$U_+UD_+ = UD_+, \quad D_+DD_+ = D_+D, \quad U_+DU_+ = DU_+$$
$$\text{and} \quad U_+DD_+ = U_+D + DD_+ - D.$$

$$UD_+D = UD_+, \quad UU_+U = U_+U, \quad DU_+D = DU_+$$
$$\text{and} \quad UU_+D = UU_+ + U_+D - U_+.$$

Step 3. Using $UD_+U = 0, UD_+D = UD_+, UU_+D = UU_+ + U_+D - U_+$ and $U_+DU_+ = DU_+$ from Step 2, we compute:

$$s_3 := \frac{1}{(t+1)(t-1)^2} U\left[tr_{(1,1)} - tr_{(1,t)} - r_{(t,1)} + r_{(t,t)}\right] = U_+D - DU_+.$$

This gives the relation

$$U_+D = DU_+.$$

Step 4. Using the relations $D_+UD_+ = 0$ and $D_+UU_+ = 0$ from Step 2 and the relation $U_+D = DU_+$ from Step 3, we compute:

$$s_4 := \frac{1}{(t+1)(t-1)^2}\left[t^2r_{(1,1)} - r_{(1,t)} - t^2r_{(t,1)} + r_{(t,t)}\right]D_+ = -UD_+.$$

This gives the relation

$$UD_+ = 0.$$

Step 5. Using the relations $UU_+U = U_+U$ from Step 2, the relation $U_+D = DU_+$ from Step 3 and the relation $UD_+ = 0$ from Step 4, we compute:

$$s_5 := \frac{1}{(t-1)^2}\left[t^2r_{(1,1)} - tr_{(1,t)} - tr_{(t,1)} + r_{(t,t)}\right]D_+ = U_+U - U_+UU_+$$

This gives the relation

$$U_+UU_+ = U_+U.$$

Furthermore, using the relation $D_+DD_+ = D_+D$ from Step 2, the relation $U_+D = DU_+$ from Step 3 and the relation $UD_+ = 0$ from Step 4, we compute:

$$s_6 := \frac{1}{(t-1)^2}\left[tr_{(1,1)} - r_{(1,t)} - tr_{(t,1)} + r_{(t,t)}\right]D_+ = -t(DD_+D - D_+D).$$

Under the assumption t invertible, this gives

$$DD_+D = D_+D.$$

Step 6. Using the relations $DD_+D = D_+DD_+$ and $UU_+U = U_+UU_+$ following from Step 2 and Step 5, the relation $U_+D = DU_+$ from Step 3, and the relation $UD_+ = 0$ from Step 4, we see that that the first, second, fifth and sixth summands in $r_{(a,b)}$ are zero. Moreover, using again $U_+D = DU_+$ and $UD_+ = 0$:

$$r_{(t,1)} = tDD_+U + D_+UU_+ - tDD_+ - UU_+ + tD + U_+.$$

We compute, using the assumptions that $t+1$ and t are invertible and the quadratic relations found in previous steps:

$$s_7 \coloneqq -\frac{1}{t(t+1)}\Big[r_{(t,1)}U_+ - tDr_{(t,1)} - U_+r_{(t,1)} + (t+1)Ur_{(t,1)}\Big] = D_+U - D_+ - U + 1.$$

This gives the last relation $D_+U = D_+ + U - 1$, which concludes the proof in the case $t \neq 0$.

7.1.3. *Case (2).* When $t = 0$, the relations $r_{(a,b)}$ reduce as follows:

$$\begin{aligned}
v_1 \coloneqq{}& & r_{(t,t)} &= UU_+U - U_+UU_+\\
v_2 \coloneqq{}& & r_{(t,t)} - r_{(1,t)} &= DU_+U - U_+UD_+\\
v_3 \coloneqq{}& & r_{(t,t)} - r_{(t,1)} &= UD_+U + UU_+D - U_+DU_+ - D_+UU_+\\
& & &\quad + UD_+D - D_+DU_+\\
v_4 \coloneqq{}& r_{(1,1)} + r_{(1,t)} + r_{(t,1)} - r_{(t,t)} & &= DD_+U + DU_+D - U_+DD_+ - D_+UD_+\\
& & &\quad + UD_+D - D_+DU_+ - DD_+D + D_+DD_+
\end{aligned}$$

Step 1. We compute:

$$\begin{aligned}
s_1 &\coloneqq U_+v_3 + v_1D + v_2U + U_+v_2D + v_3U + v_3D + D_+v_1D + D_+v_2D - v_1 - 2v_3\\
&= DU_+U - U_+DU_+ + DU_+ - UD_+.
\end{aligned}$$

This gives a relation between words of length at most three.

Step 2. We compute:

$$s_2 \coloneqq Ds_1U_+ - s_1U_+ - Ds_1 + s_1 = -UD_+.$$

This gives the relation

$$UD_+ = 0.$$

Step 3. With the relation from Step 2, we have that $v_2 = DU_+U$. Multiplying by U on the left gives $UU_+U - U_+U$. Together with v_1, it gives the relations

$$UU_+U = U_+UU_+ = U_+U.$$

Step 4. It follows from Step 3 that $UU_+D = U_+D + UU_+ - U_+$, computing $(UU_+U - U_+U)D$. With this and Step 2, v_3 simplifies as:

$$v_3 = -U_+DU_+ - D_+UU_+ - D_+DU_+ + U_+D + UU_+ - U_+$$

Computing $U_+v_3 - v_3 - Uv_3$ leads to the relation

$$DU_+ = U_+D.$$

Note that thanks to Remark 7.4, it also follows that $D_+DD_+ = D_+D$.

Step 5. With the relation $DU_+ = U_+D$ from Step 4, v_3 further simplifies as:

$$v_3 = -D_+UU_+ + UU_+ - U_+$$

Using $UD_+ = 0$, computing $v_3D_+ - v_3$ leads to the relation

$$D_+U = D_+ + U - 1.$$

Step 6. Using the relations found in previous steps, the relation v_4 simplifies as follows:

$$v_4 = -DD_+D + D_+D.$$

This gives the remaining relation $DD_+D = D_+D$, and concludes.

7.2. **The structure as augmented algebra.** The aim of this section is to describe the quotient of $\mathcal{LH}_n$ by the two-sided ideal generated by

$$\chi^{(j+1)} := (\sigma_1 - \rho_1)\cdots(\sigma_j - \rho_j)$$

for $1 \leq j \leq n$. If one assumes that $t \neq \pm 1$, then via Theorem 7.1 this is equivalent to describing the quotient of $\widetilde{\mathcal{LH}}_n$ by the element $D_1 \cdots D_j$. It is the latter that we do in this section:

Proposition 7.5. *The map*

$$\pi_n : \widetilde{\mathcal{LH}}_n \to \mathbb{Z} : \begin{cases} U_i \mapsto 1 \\ D_i \mapsto 0 \end{cases}$$

is a $\mathbb{Z}$-algebra morphism with $\ker(\pi_n) = (D_1 \cdots D_j)$ for any $1 \leq j \leq n$.

Proof. Using a direct verification it is easily verified that π_n is well-defined, i.e. that the defining relations in Definition 1.4 of $\widetilde{\mathcal{LH}}_n$ are satisfied under π_n.

Note that $\ker(\pi_n) = (D_1, \ldots, D_n)$ as the quotient map $\widetilde{\mathcal{LH}}_n \twoheadrightarrow \widetilde{\mathcal{LH}}_n/(D_1, \ldots, D_n)$ maps U_i to 1 by the relation $U_iD_i = U_i + D_i - 1$, see (7). Therefore, this quotient map agrees with π_n and it remains to prove that $(D_1, \ldots, D_n) = (D_1 . \cdots . D_j)$ for any $1 \leq j \leq n$. Equivalently, we prove that $D_i \equiv 0$ in $\widetilde{\mathcal{LH}}_n/(D_1 \cdots D_j)$ for any $1 \leq i \leq n-1$.

We will use $\equiv$ to emphasize that we are working with the quotient $\widetilde{\mathcal{LH}}_n/(D_1 \cdots D_j)$. As $0 \equiv D_1 \cdots D_j$, also $U_1D_1 \cdots D_m \equiv 0$ for $j \leq m$. Consequently, if $m > 1$ and using the relations for U_1D_1 and U_1D_2:

$$\underbrace{U_1D_2}_{=0} D_3 \cdots D_m + \underbrace{D_1 \cdots D_m}_{=0} - D_2 \cdots D_m \equiv 0.$$

Thus we obtained that $D_2 \cdots D_m \equiv 0$. Continuing iteratively, we obtain that $D_m \equiv 0$ and hence $D_j \equiv \cdots \equiv D_n \equiv 0$.

For the variables D_i with $i < j$ we consider $D_i \cdots D_jU_{j-1} \equiv 0$. For that word, the defining relation (8) yields

$$\underbrace{D_i \cdots D_j}_{=0} + D_i \cdots D_{j-2} \underbrace{D_{j-1}U_{j-1}}_{=0} - D_i \cdots D_{j-1} \equiv 0.$$

This time, continuing iteratively with $D_i \cdots D_{j-1} \equiv 0$, we obtain that $D_i \equiv 0$, as desired. $\square$

Remark 7.6. In [12, Conjecture 6.4] it was conjectured that the quotient would be non-trivial for $j \neq 1$ and that furthermore the values $\dim e_j \mathcal{L}\mathcal{H}_n/(\chi^{(j+1)})e_i$, with the e_j a system of orthogonal idempotents of $\mathcal{H}_n/(\chi^{(j+1)})$ adding up to the identity, would be given by the $j \times j$-truncation of the matrix in Corollary 6.3. Although Proposition 7.5 shows wrong, it would be interesting to know if there exist some quotients of the loop Hecke algebra that yield those truncations.

7.3. **Comparison with the Hecke-Hopf algebra.** In [2] the Hecke-Hopf algebra $\mathbf{H}(\mathrm{S}_n)$ was introduced. This Hopf $\mathbb{Z}$-algebra [2, Theorem 1.3] has the interesting property that the classical Hecke algebra embeds in it [2, Theorem 1.9]: $H_q(\mathrm{S}_n) \hookrightarrow \mathbf{H}(\mathrm{S}_n) \otimes_{\mathbb{Z}} \mathbb{Z}[q, q^{-1}]$. The algebra is defined as follows.

Definition 7.7. Let $n \geq 2$. The Hecke-Hopf algebra, denoted $\mathbf{H}(\mathrm{S}_n)$, is the $\mathbb{Z}$-algebra generated by s_i and D_i for $i = 1, \dots, n-1$ and subject to the following relations:

- $s_i^2 = 1, \quad s_i D_i + D_i s_i = s_i - 1, \quad D_i^2 = D_i$ for $1 \leq i \leq n-1$,
- $s_j s_i = s_i s_j, \; D_j s_i = s_i D_j, \; D_j D_i = D_i D_j$ for $|i - j| > 1$,
- $s_j s_i s_j = s_i s_j s_i, \; D_i s_j s_i = s_j s_i D_j, \; D_j s_i D_j = s_i D_j D_i + D_i D_j s_i + s_i D_j s_i$ for $|i-j| = 1$.

Note that, when $t - 1$ is invertible, the presentation of the loop Hecke algebra $\mathcal{L}\mathcal{H}_n$ in the generators $\{\rho_i, D_i := \frac{(\sigma_i - \rho_i)}{1-t}\}$ has quite some similarities with the Hopf-Hecke algebra. Using the presentation obtained via Theorem 7.1 we will rather compare with $\widetilde{\mathcal{L}\mathcal{H}}_n$ which is also a $\mathbb{Z}$-algebra.

Proposition 7.8. *Let $n \geq 2$. Then the map*

$$\psi_n \colon \mathbf{H}(\mathrm{S}_n) \to \widetilde{\mathcal{L}\mathcal{H}}_n \colon \begin{cases} s_i & \mapsto U_i - D_i \\ D_i & \mapsto D_i \end{cases}$$

is an epimorphism of augmented $\mathbb{Z}$-algebras. However, $\ker(\psi_n)$ *is* not *a Hopf ideal of* $\mathbf{H}(\mathrm{S}_n)$.

Remark 7.9. Unfortunately, as $\ker(\psi_n)$ is not a Hopf ideal, the Hopf structure of the Hecke-Hopf algebra cannot be transported to the loop Hecke algebra. In fact for $n = 2, 3, 4$ the loop Hecke algebra is *not a Hopf algebra*.

Indeed, in those cases $\dim_{\Bbbk}(\widetilde{\mathcal{L}\mathcal{H}}_n \otimes_{\mathbb{Z}[t]} \Bbbk) = \binom{2n-1}{n} = 3$, 10 and 35 when $n = 2$, 3 and 4 respectively. For these dimensions it is known that all Hopf algebras over an algebraically closed field $\Bbbk$ of characteristic 0 are semisimple, see [26, 25]. However, it follows from the combination of Theorem 5.7 and Theorem 6.1 that the loop Hecke algebra is not semisimple.

Thus it seems that the loop Hecke algebra shares with the Hecke algebra the fact of not being a Hopf algebra. Hence it is reasonable to ask the following.

Question 1. Does there exist a “loop Hecke-Hopf algebra”, i.e. a Hopf algebra in which the loop Hecke algebra canonically embeds?

We now proceed to the proof.

Proof of Proposition 7.8. By construction ψ_n will be an epimorphism of $\mathbb{Z}$-algebras if it is well-defined. That it is one of augmented algebras means that $\epsilon_{\mathbf{H}(\mathrm{S}_n)} = \pi_n \circ \psi_n$ where $\epsilon_{\mathbf{H}(\mathrm{S}_n)}$ is the counit $\mathbf{H}(\mathrm{S}_n)$ and π_n is defined in Proposition 7.5. Recall that by definition $\epsilon_{\mathbf{H}(\mathrm{S}_n)}(s_i) = 1$ and $\epsilon_{\mathbf{H}(\mathrm{S}_n)}(D_i) = 0$ and so we indeed see that ψ_n commutes with the augmentation.

Next, we verify the well-definedness.

The relations $\psi_n(s_i^2) = 1$ and $\psi_n(s_iD_i + D_is_i) = \psi_n(s_i - 1)$ follow via a direct computation using the same-label relations (7).

The relations for $|i-j| > 1$ follow from the interchange relations (11).

The relation $\psi_n(s_js_is_j) = \psi_n(s_is_js_i)$ follows from direct computation, analogous to how the relation $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$ in $\mathcal{LH}_n$ (see (2)) followed from relations in $\widetilde{\mathcal{LH}_n}$ in the proof of Theorem 7.1. Using quadratic relations (7) and (8) and considering the cases $j = i \pm 1$ separately, the relation $\psi_n(D_is_js_i) = \psi_n(s_js_iD_j)$ reduces to:

$$\begin{aligned} -D_iU_{i+1} - D_iD_{i+1} + D_i + D_iD_{i+1}D_i &= -D_iU_{i+1} - D_iD_{i+1} + D_i + D_{i+1}D_iD_{i+1} \\ \text{and} \quad U_iU_{i+1} - U_{i+1} + D_{i+1}D_iD_{i+1} &= U_iU_{i+1} + D_iU_{i+1} - U_{i+1} - D_iU_{i+1} + D_iD_{i+1}D_i. \end{aligned}$$

Both hold thanks to $D_iD_{i+1}D_i = D_{i+1}D_iD_{i+1}$. Finally, we check the relation $\psi_n(D_js_iD_j) = \psi_n(s_iD_jD_i + D_iD_js_i + s_iD_js_i)$. On the one hand $\psi_n(D_js_iD_j) = D_jU_iD_j - D_jD_iD_j$, and on the other hand a direct computation gives

$$\psi_n(s_iD_jD_i + D_iD_js_i + s_iD_js_i) = U_iD_jU_i - D_iD_jD_i.$$

Considering the cases $j = i \pm 1$ separately, it is a direct computation that $U_iD_jU_i = D_jU_iD_j$.

Finally, we prove that $\ker(\psi_n)$ is not a Hopf-ideal. Note that $D_is_i + D_i \in \ker(\psi_n)$. To show that $\ker(\psi_n)$ is not a Hopf ideal it suffices to show that[12] $S((s_i + D_i)D_{i+1}) \notin \ker(\psi_n)$. To do so, recall that the antipode is defined by $S(s_i) = s_i$ and $S(D_i) = -s_iD_i$. Hence

$$S((s_i + D_i)D_{i+1}) = s_i(1 - D_i)D_{i+1}.$$

Now, using (7) we find:

$$\Psi_n(s_i(1-D_i)D_{i+1}) = (U_i - D_i)(1 - D_i)D_{i+1} = U_iD_{i+1} - U_iD_iD_{i+1} = -D_iD_{i+1} + D_{i+1}.$$

The latter is non-zero as the monomials are $\widetilde{\mathcal{LH}_n}$-reduced words and hence linearly independent by Theorem 2.1. □

Appendix A. Confluence of critical branchings

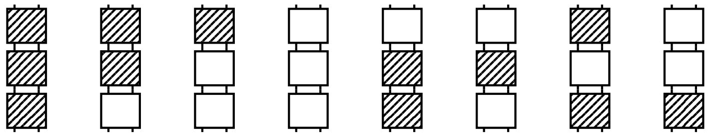
In this appendix, we finish the proof of Theorem 2.1 by showing that the higher linear rewriting system described in Figure 1 (see Figure 2 for the diagrammatic notation) critically confluate.

We enumerate critical branchings by first considering critical branchings involving a same-label rewriting step (Section A.1), and then all the remaining critical branchings (Section A.2).

A.1. **Same-label rewriting steps and others.**

[12] Alternatively one can verify that it is not a coideal, hence $\ker(\psi_n)$ is also not a bialgebra ideal.

A.1.1. *Same-label rewriting steps and same-label rewriting steps.* We consider branchings whose branches are of type one of the same-label rewriting steps:

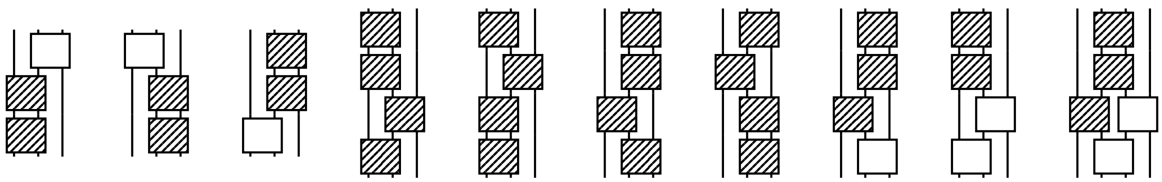


The first four branchings are straightforward. The last four come down to the fact that

$$U \cdot (U + D - 1) \xrightarrow{*}_{\mathsf{R}} UD \text{ and } D \cdot (U + D - 1) \xrightarrow{*}_{\mathsf{R}} 0,$$

and vice-versa for the multiplication on the right.

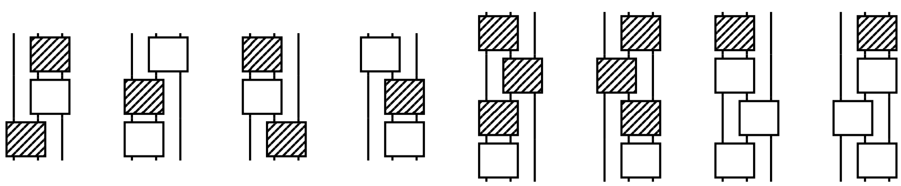
A.1.2. $DD \to \ldots$ *and others.* We consider branchings with one branch of type $DD \to \ldots$ and the other branch of type one of the distinct-label or additional rewriting steps:



Their confluence is relatively straightforward. The confluence of the third branching comes down to the fact that

$$D_+(D_+ + U - 1) \xrightarrow{*}_{\mathsf{R}} D_+ + U - 1.$$

A.1.3. $DU \to \ldots$ *and others.* We consider branchings with one branch of type $DU \to \ldots$ and the other branch of type one of the distinct-label or additional rewriting steps:



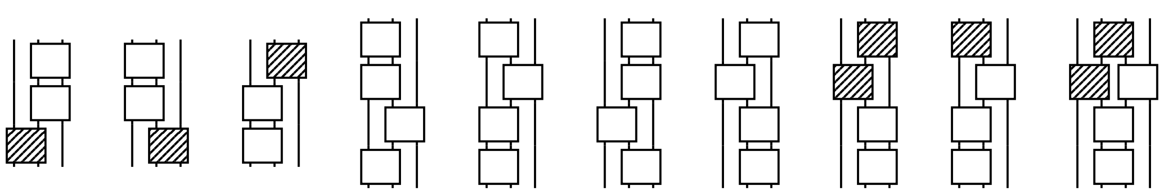
Since $DU \to \ldots$ rewrites to zero, it suffices to check that the other branch rewrites to zero.

Confluence of the first (resp. second) branching uses the first (resp. second) additional rewriting step. The same holds for the sixth and seventh branchings.

Confluence of the third and fourth branchings is immediate, as the other branch rewrites to zero.

Confluence of the fifth and eighth branchings is straightforward.

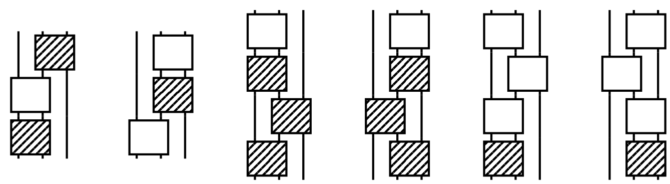
A.1.4. $UU \to \ldots$ *and others.* We consider branchings with one branch of type $UU \to \ldots$ and the other branch of type one of the distinct-label or additional rewriting steps:



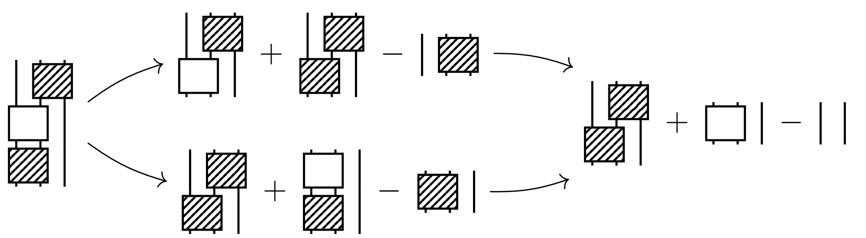
Their confluence is relatively straightforward and similar to Section A.1.2. The confluence of the third branching comes down to the fact that

$$(D_+ + U - 1)U \xrightarrow{*}_{\mathsf{R}} D_+ + U - 1.$$

A.1.5. $UD \to \ldots$ *and others.* We consider branchings with one branch of type $UD \to \ldots$ and the other branch of type one of the distinct-label rewriting steps:

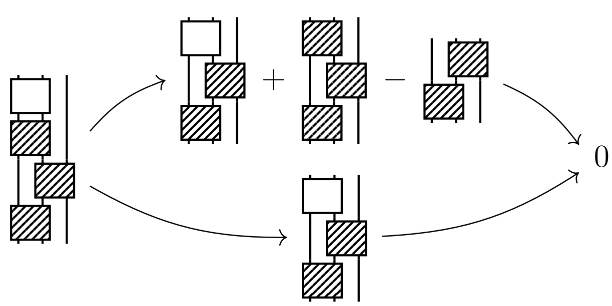


The confluence of the first branching is given below:

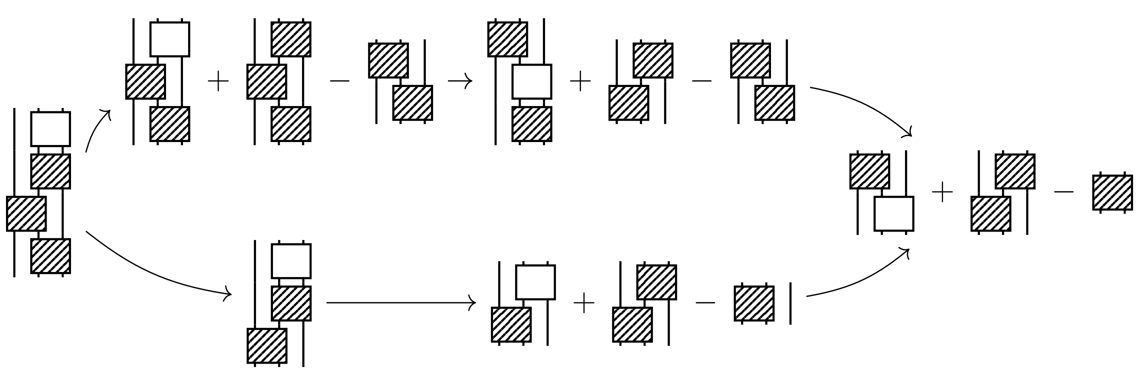


The confluence of the second branching is similar.

The confluence of the third branching is given below:

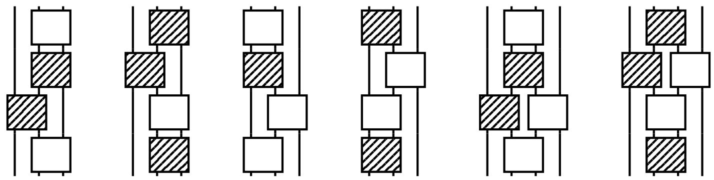


The confluence of the fourth branching is given below:



The confluence of the fifth and sixth branchings is obtained similarly.

Next, we consider branchings with one branch of type $UD \to \ldots$ and the other branch of type one of the additional rewriting steps:

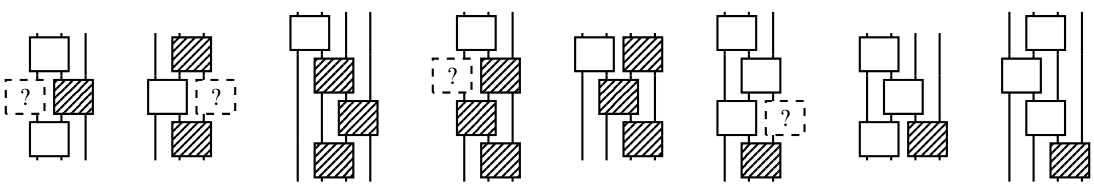


The other branch always rewrite to zero. We leave it to the reader to check that the branch $UD \to \ldots$ also rewrites to zero.

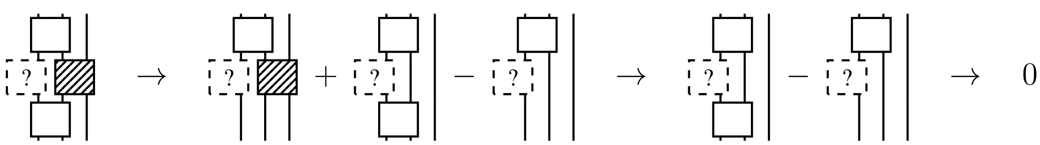
A.2. **Distinct-label rewriting steps and others.**

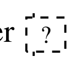
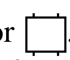
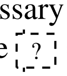
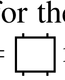
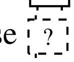
A.2.1. $U_+D \to \ldots$ *and others.* This was done in Section 2.3.

A.2.2. $UD_+ \to \ldots$ *and others.* We find the following list of critical branchings with the remaining distinct-label rewriting steps:



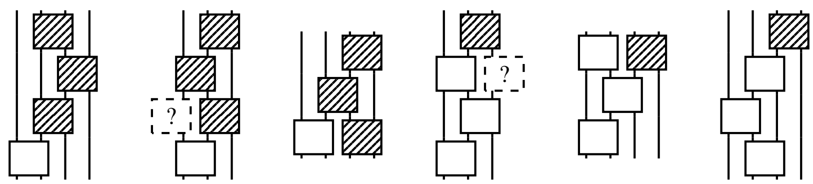
The branch $UD_+ \to \ldots$ rewrites into zero: to show confluence, we must check that the other branch rewrites to zero. For the first branching, we have:



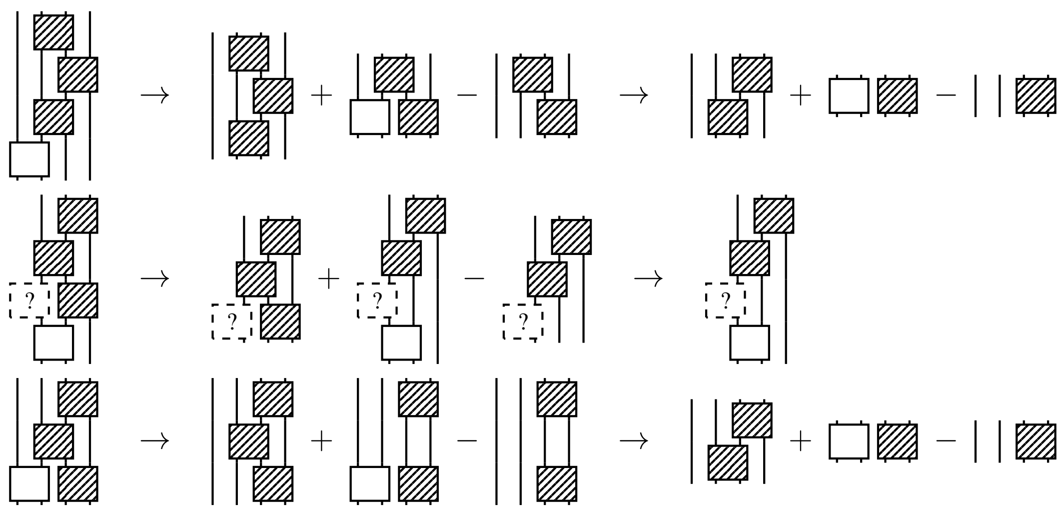
The last step requires a case by case analysis, depending on whether  is ,  or . The second branching is analogous. A similar case by case analysis is necessary for the fourth and sixth branchings. Note that confluence of the fourth branching in the case  $=$  requires one the additional rewriting step, and similarly for the sixth branching in the case  $=$ . The remaining branchings are straightforward.

Since both $UD_+ \to \ldots$ and the additional rewriting steps rewrite to zero, any branching between them is automatically confluent; hence we don't bother classifying these critical branchings.

A.2.3. $D_+U \to \ldots$ *and others.* We consider branchings involving a branch of type $D_+U \to \ldots$ and branch of type one of the remaining distinct-label rewriting steps.

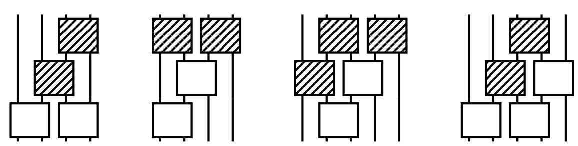


We compute the $D_+U \to \ldots$ branch of the first three branchings:



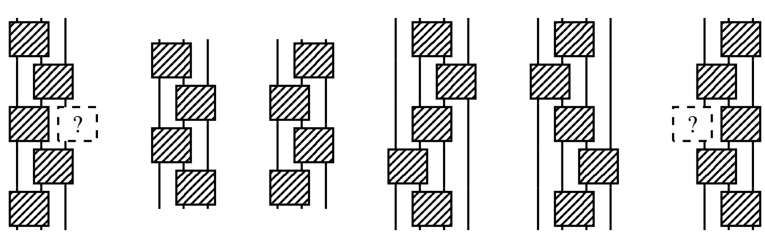
One checks that this confluates with the other branch. The last three branchings are analogous.

We find the following list of critical branchings with the additional rewriting steps:



One checks that both branches rewrite to zero.

A.2.4. *Three-term rewriting steps and others.* We consider branchings involving one the three-term rewriting steps in the set of distinct-label rewriting steps. First, consider the case where both branches are of this type. The situation is symmetric in the D's and U's, so we only consider branchings in D's:



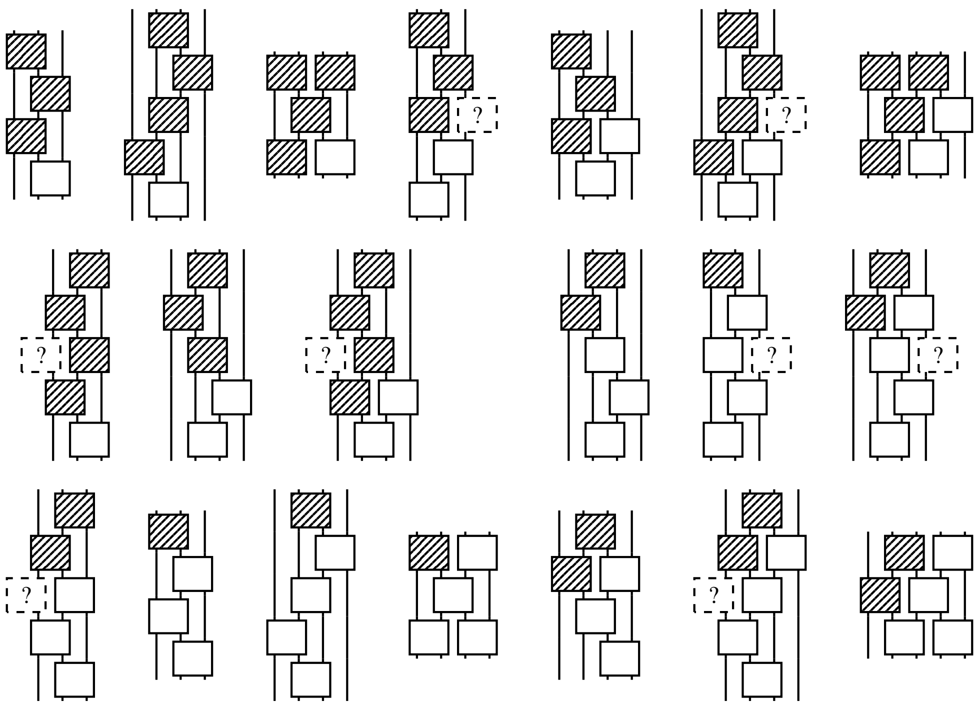
Consider the first branching. If $\boxed{?} = \square$, we can use the rewriting step $U_+D \to DU_+$ to slide this U out of the diagram, and we recover the case $\boxed{?} = |$. Moreover, the last branching rewrites into zero when $\boxed{?} = \square$, irrespective of the branch.

When $\boxed{?} = \blacksquare$, the first and last branchings rewrite into $D_{++}D_+D$, irrespective of the branch. The same holds for the fourth branching.

The fifth branching rewrites into $D_+DD_{++}D_+$, irrespective of the branch.

Finally, when $\boxed{?} = |$, the first and last branchings rewrite into D_+D, irrespective of the branch. The same holds for the second and third branchings.

Consider then branchings where one branch is a three-term rewriting step and the other branching is one the additional rewriting steps:



Since additional rewriting steps rewrite to zero, it suffices to check that the other branch rewrites to zero. This is straightforward for branchings in the first row, as the additional rewriting step can still be applied after applying the three-term rewriting step. The same is true for branchings in the third row.

Consider the first branching of the second row. The case $\boxed{?} = \square$ rewrites to zero, and when $\boxed{?} = \blacksquare$ or $\boxed{?} = |$, we can rewrite until an additional rewriting step can be applied. Similar arguments apply to the remaining branchings in the second row.

A.2.5. *additional rewriting steps and additional rewriting steps.* Both branches rewrite to zero, so any such branching is trivially confluent.

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